Modeling in the Frequency Domain

SOLUTIONS TO CASE STUDIES CHALLENGES

Antenna Control: Transfer Functions

Finding each transfer function:

Pot: \( \frac{V_i(s)}{\theta_i(s)} = \frac{10}{\pi} \);

Pre-Amp: \( \frac{V_i(s)}{V_p(s)} = K; \)

Power Amp: \( \frac{E_p(s)}{V_p(s)} = \frac{150}{s + 150} \)

Motor: \( J_m = 0.05 + 5\left( \frac{50}{250} \right)^2 = 0.25 \)

\( D_m = 0.01 + 3\left( \frac{50}{250} \right)^2 = 0.13 \)

\( \frac{K_a}{R_a} = \frac{1}{5} \)

\( \frac{K_aK_b}{R_a} = \frac{1}{5} \)

Therefore:

\( \frac{\theta_{m_a}(s)}{E_a(s)} = \frac{K_a}{s(s + 1)(D_m + \frac{K_aK_b}{R_a})} = \frac{0.8}{s(s + 1.32)} \)

And:

\( \frac{\theta_o(s)}{E_a(s)} = \frac{1}{5} \frac{\theta_{m_a}(s)}{E_a(s)} = \frac{0.16}{s(s + 1.32)} \)

Transfer Function of a Nonlinear Electrical Network

Writing the differential equation,

\( \frac{d(i_o + \delta i)}{dt} + 2(i_o + \delta i)^2 - 5 = v(t) \) . Linearizing \( i^2 \) about \( i_o \),

\( (i_o + \delta i)^2 - i_o^2 = 2i_o \delta i = 2i_o \delta i. \) Thus, \( (i_o + \delta i)^2 = i_o^2 + 2i_o \delta i. \)
Substituting into the differential equation yields, \( \frac{di}{dt} + 2i_0^2 + 4i_0\delta i - 5 = v(t) \). But, the resistor voltage equals the battery voltage at equilibrium when the supply voltage is zero since the voltage across the inductor is zero at dc. Hence, \( 2i_0^2 = 5 \), or \( i_0 = 1.58 \). Substituting into the linearized differential equation, \( \frac{\delta i}{dt} + 6.32\delta i = v(t) \). Converting to a transfer function, \( \frac{\delta i(s)}{V(s)} = \frac{1}{s + 6.32} \). Using the linearized i about \( i_0 \), and the fact that \( v_r(t) \) is 5 volts at equilibrium, the linearized \( v_r(t) = 2i^2 = 2(i_0^2 + \delta i)^2 = 2(i_0^2 + 2i_0\delta i) = 5 + 6.32\delta i \). For excursions away from equilibrium, \( v_r(t) - 5 = 6.32\delta i = \delta v_r(t) \).

Therefore, multiplying the transfer function by 6.32, yields, \( \frac{\delta V_r(s)}{V(s)} = \frac{6.32}{s + 6.32} \) as the transfer function about \( v(t) = 0 \).

ANSWERS TO REVIEW QUESTIONS

1. Transfer function
2. Linear time-invariant
3. Laplace
4. \( G(s) = \frac{C(s)}{R(s)} \), where \( c(t) \) is the output and \( r(t) \) is the input.
5. Initial conditions are zero
6. Equations of motion
7. Free body diagram
8. There are direct analogies between the electrical variables and components and the mechanical variables and components.
9. Mechanical advantage for rotating systems
10. Armature inertia, armature damping, load inertia, load damping
11. Multiply the transfer function by the gear ratio relating armature position to load position.
12. (1) Recognize the nonlinear component, (2) Write the nonlinear differential equation, (3) Select the equilibrium solution, (4) Linearize the nonlinear differential equation, (5) Take the Laplace transform of the linearized differential equation, (6) Find the transfer function.

SOLUTIONS TO PROBLEMS

1.

a. \( F(s) = \int_0^\infty e^{-st} dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = \frac{1}{s} \)

b. \( F(s) = \int_0^\infty te^{-st} dt = \frac{e^{-st}}{s^2}(-st - 1)\bigg|_0^\infty = \frac{-(st + 1)}{s^2 e^{st}}\bigg|_0^\infty \)
Using L'Hopital's Rule

\[ F(s)\bigg|_{s \to \infty} = \frac{-s}{s^3 e^s} \bigg|_{s \to \infty} = 0. \] Therefore, \( F(s) = \frac{1}{s^2} \).

c. \( F(s) = \int_0^\infty \sin \omega t \ e^{-st} \, dt = \frac{e^{-st}}{s^2 + \omega^2} (-s \sin \omega t - \omega \cos \omega t) \bigg|_0^\infty = \frac{\omega}{s^2 + \omega^2} \)

d. \( F(s) = \int_0^\infty \cos \omega t \ e^{-st} \, dt = \frac{e^{-st}}{s^2 + \omega^2} (-s \cos \omega t + \omega \sin \omega t) \bigg|_0^\infty = \frac{s}{s^2 + \omega^2} \)

2.

a. Using the frequency shift theorem and the Laplace transform of \( \sin \omega t \), \( F(s) = \frac{\omega}{(s+a)^2 + \omega^2} \).

b. Using the frequency shift theorem and the Laplace transform of \( \cos \omega t \), \( F(s) = \frac{(s+a)}{(s+a)^2 + \omega^2} \).

c. Using the integration theorem, and successively integrating \( u(t) \) three times, \( \int dt = t; \int t \, dt = \frac{t^2}{2} \):
   \[ \int \frac{t^3}{2} \, dt = \frac{3}{6}, \] the Laplace transform of \( t^3 u(t) \), \( F(s) = \frac{6}{s^4} \).

3.

a. Taking the sum of the voltages around the loop and assuming zero initial conditions yields:

\[ Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int_0^t i(\tau) \, d\tau = v(t) \]

b. Applying Laplace transform and solving for \( I(s)/V(s) \) gives:

\[ \frac{1}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} = \frac{1}{L\left(s + \frac{R}{L} + \frac{1}{LCs}\right)} \]

Substituting the values of \( R, L, \) and \( LC \), we have:

\[ \frac{I(s)}{V(s)} = \frac{2}{(s + 2 + \frac{16}{s})} = \frac{2s}{s^2 + 2s + 16} \]
Solving for \( I(s) \) and noting that \( V(s) = 1/s \), we get:

\[
I(s) = \frac{2}{s^2 + 2s + 16}
\]

Observing that the denominator has complex roots, we re-write the above equation as:

\[
I(s) = \frac{2}{(s + 1)^2 + (\sqrt{15})^2}
\]

Applying the frequency shift theorem to the Laplace transform of \( \sin(\omega u(t)) \), we find that the transform for

\[
f(t) = e^{-at} \sin(\omega t)
\]

is

\[
F(s) = \frac{\omega}{(s + a)^2 + \omega^2}.
\]

Comparing \( F(s) \) to \( I(s) \), we conclude that in the latter: \( a = 1 \) and \( \omega = \sqrt{15} \). Thus, the current, \( i(t) \), may be given by:

\[
i(t) = \frac{2}{15} \sqrt{15} \ e^{-t} \sin(\sqrt{15} t)
\]
4. a. The Laplace transform of the differential equation, assuming zero initial conditions,

is, \((s+7)X(s) = \frac{5s}{s^2 + 2^2}\). Solving for \(X(s)\) and expanding by partial fractions,

\[
\frac{5s}{(s+7)(s^2 + 4)} = \frac{35}{53} \frac{1}{s+7} + \frac{5}{53} \frac{7s+4}{s^2 + 4}
\]

Or,

\[
\frac{5s}{(s+7)(s^2 + 4)} = \frac{35}{53} \frac{1}{s+7} + \frac{5}{53} \frac{7s+2\sqrt{2}}{s^2 + 4}
\]

Taking the inverse Laplace transform, \(x(t) = -\frac{35}{53} e^{-7t} + (\frac{35}{53} \cos 2t + \frac{10}{53} \sin 2t)\).

b. The Laplace transform of the differential equation, assuming zero initial conditions, is,

\((s^2+6s+8)X(s) = \frac{15}{s^2 + 9}\).

Solving for \(X(s)\)

\[X(s) = \frac{15}{(s^2 + 9)(s^2 + 6s + 8)}\]

and expanding by partial fractions,

\[X(s) = \frac{6s + \frac{1}{\sqrt{9}}}{65} \frac{\sqrt{9}}{s^2 + 9} - \frac{3}{10} \frac{1}{s + 4} + \frac{15}{26} \frac{1}{s + 2}\]

Taking the inverse Laplace transform,

\[x(t) = -\frac{18}{65} \cos(3t) - \frac{1}{65} \sin(3t) - \frac{3}{10} e^{-4t} + \frac{15}{26} e^{-2t}\]

c. The Laplace transform of the differential equation is, assuming zero initial conditions,

\((s^2+8s+25)x(s) = \frac{10}{s}\). Solving for \(X(s)\)

\[X(s) = \frac{10}{s(s^2 + 8s + 25)}\]

and expanding by partial fractions,

\[X(s) = \frac{2}{5} \frac{1}{s} + \frac{2}{5} \frac{1(s+4) + \frac{4}{\sqrt{9}}}{s + 4^2 + 9}\]

Taking the inverse Laplace transform,

\[x(t) = \frac{2}{5} e^{-\frac{t}{5}} \left(\frac{8}{15} \sin(3t) + \frac{2}{5} \cos(3t)\right)\]
5. a. Taking the Laplace transform with initial conditions, \( s^2X(s) - 4s + 4 + 2sX(s) - 8 + 2X(s) = \frac{2}{s^2 + 2^2} \).

Solving for \( X(s) \),
\[
X(s) = \frac{4s^3 + 4s^2 + 16s + 18}{(s^2 + 4)(s^2 + 2s + 2)}.
\]

Expanding by partial fractions
\[
X(s) = \frac{1}{5} \left( \frac{1}{s^2 + 2^2} + \frac{1}{s} \right) \frac{21(s + 1) + 2}{(s + 1)^2 + 1}
\]

Therefore, \( x(t) = \frac{1}{5} \left[ 21e^{-t} \cos t + \frac{2}{21} e^{-t} \sin t - \frac{1}{2} \sin 2t - \cos 2t \right] \).

b. Taking the Laplace transform with initial conditions, \( s^2X(s) - 4s + 1 + 2sX(s) - 8 + 8X(s) = \frac{5}{s^2 + 1} + \frac{1}{s^2} \).

Solving for \( X(s) \),
\[
X(s) = \frac{4s^4 + 17s^3 + 23s^2 + s + 2}{s^2(s + 1)^2(s + 2)}
\]

\[
X(s) = \frac{1}{s^2} - \frac{2}{s} + \frac{11}{(s + 1)^2} + \frac{1}{(s + 1)} + \frac{5}{(s + 2)}
\]

Therefore \( x(t) = t - 2 + \frac{1}{11}t e^{-t} + e^{-t} + 5e^{-2t} \).

c. Taking the Laplace transform with initial conditions, \( s^2X(s) - s - 2 + 4X(s) = \frac{2}{s^3} \).

Solving for \( X(s) \),
\[
X(s) = \frac{2s^4 + 3s^3 + 2}{s^3(s^2 + 4)}
\]

\[
\frac{17}{8} \frac{s + 3/2}{s^2 + 4} + \frac{1/2}{s^3} - \frac{1/8}{s}
\]

Therefore \( x(t) = \frac{17}{8} \cos 2t + \frac{3}{2} \sin 2t + \frac{1}{4} t^2 - \frac{1}{8} \).

6. **Program:**

```matlab
syms t
'\( a \)'
theta=45*pi/180
f=8*t^2*cos(3*t+theta);
pretty(f)
F=laplace(f);
F=simple(F);
pretty(F)
'\( b \)'
theta=60*pi/180
f=3*t*exp(-2*t)*sin(4*t+theta);
pretty(f)
```
\begin{verbatim}
F=laplace(f);
F=simple(F);
pretty(F)

Computer response:

ans =
a
theta =
0.7854
\[
\begin{align*}
8 t \cos \theta + 3 t \\
\frac{1}{4}
\end{align*}
\]
\[
\frac{\sqrt{8}}{2}
\frac{2}{(s + 3) (s - 12 s + 9)}
\]
\[
\frac{3}{2}
\frac{3}{s + 9}
\]

ans =
b
theta =
1.0472
\[
\begin{align*}
3 t \sin \theta + 4 t \exp(-2 t)
\end{align*}
\]
\[
\frac{\sqrt{3}}{3}
\frac{1/2}{2}
\frac{1/2}{2}
\]
\[
\frac{12 s + 6}{3}
\frac{1}{s - 18}
\frac{3}{3} + \frac{s}{2} + 24
\]
\[
\frac{2}{s + 4 s + 20}
\]
\end{verbatim}

7. Program:
syms s
'a'
G = (s^2 + 3*s + 10)*(s+5)/[(s+3)*(s+4)*(s^2+2*s+100)];
pretty(G)
g = ilaplace(G);
pretty(g)
'b'
G = (s^3 + 4*s^2 + 2*s + 6)/[(s+8)*(s^2+8*s+3)*(s^2+5*s+7)];
pretty(G)
g = ilaplace(G);
pretty(g)

Computer response:
ans =

\begin{align*}
a & \frac{2}{(s + 5) (s + 3 s + 10)} \hfill \\
& \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
& \frac{5203}{103} \frac{7}{54} \frac{5723}{5562} \\
& \frac{20 \exp(-3 t)}{5203} \frac{7 \exp(-4 t)}{5723} \\
& \frac{1}{15587} \\
& \end{align*}

ans =

\begin{align*}
b & \frac{3}{s + 4 s + 2 s + 6} \hfill \\
& \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \\
& \frac{1}{1199} \frac{1}{1199} \frac{1}{1199} \frac{1}{1199} \frac{1}{1199} \\
& \frac{4262}{15587} \frac{13}{15587} \frac{4262}{15587} \frac{13}{15587} \frac{4262}{15587} \\
& \frac{37}{15587} \frac{37}{15587} \frac{37}{15587} \frac{37}{15587} \frac{37}{15587} \\
& \end{align*}
8. The Laplace transform of the differential equation, assuming zero initial conditions, is,

$$(s^3 + 3s^2 + 5s + 1)Y(s) = (s^3 + 4s^2 + 6s + 8)X(s).$$

Solving for the transfer function,

$$\frac{Y(s)}{X(s)} = \frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + 5s + 1}.
$$

9. a. Cross multiplying, $(s^2 + 5s + 10)X(s) = 7F(s)$.

Taking the inverse Laplace transform,

$$\frac{d^2x}{dt^2} + 7 \frac{dx}{dt} + 10x = 7f.$$

b. Cross multiplying after expanding the denominator, $(s^2 + 21s + 110)X(s) = 15F(s)$.

Taking the inverse Laplace transform,

$$\frac{d^2x}{dt^2} + 21 \frac{dx}{dt} + 110x = 15f.$$

c. Cross multiplying, $(s^3 + 11s^2 + 12s + 18)X(s) = (s + 3)F(s)$.

Taking the inverse Laplace transform,

$$\frac{d^3x}{dt^3} + 11 \frac{d^2x}{dt^2} + 12 \frac{dx}{dt} + 18x = \frac{df}{dt} + 3f.$$

10. The transfer function is

$$\frac{C(s)}{R(s)} = \frac{s^5 + 2s^4 + 4s^3 + s^2 + 4}{s^6 + 7s^5 + 3s^4 + 2s^3 + s^2 + 5}.$$

Cross multiplying, $(s^6 + 7s^5 + 3s^4 + 2s^3 + s^2 + 5)C(s) = (s^5 + 2s^4 + 4s^3 + s^2 + 4)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^6c}{dt^6} + 7 \frac{d^5c}{dt^5} + 3 \frac{d^4c}{dt^4} + 2 \frac{d^3c}{dt^3} + \frac{d^2c}{dt^2} + 5c = \frac{d^5r}{dt^5} + 2 \frac{d^4r}{dt^4} + 4 \frac{d^3r}{dt^3} + \frac{d^2r}{dt^2} + 4r.$$

11. The transfer function is

$$\frac{C(s)}{R(s)} = \frac{s^5 + 2s^4 + 5s^3 + s + 1}{s^5 + 3s^4 + 2s^3 + 4s^2 + 5s + 2}.$$

Cross multiplying, $(s^5 + 3s^4 + 2s^3 + 4s^2 + 5s + 2)C(s) = (s^4 + 2s^3 + 5s^2 + s + 1)R(s)$.

Taking the inverse Laplace transform assuming zero initial conditions,

$$\frac{d^5c}{dt^5} + 3 \frac{d^4c}{dt^4} + 2 \frac{d^3c}{dt^3} + 4 \frac{d^2c}{dt^2} + \frac{dc}{dt} + 2c = \frac{d^4r}{dt^4} + 2 \frac{d^3r}{dt^3} + 5 \frac{d^2r}{dt^2} + \frac{dr}{dt} + r.$$

Chapter 2: Modeling in the Frequency Domain

Substituting \( r(t) = t^3 \),

\[
\begin{align*}
\frac{d^3 c}{dt^3} + 3 \frac{d^4 c}{dt^4} + 2 \frac{d^5 c}{dt^5} + 4 \frac{d^2 c}{dt^2} + 5 \frac{dc}{dt} + 2c &= 18\delta(t) + (36 + 90t + 9t^2 + 3t^3) u(t).
\end{align*}
\]

12.

Taking Laplace transform of the differential equation:

\[
s^2X(s) - s + 1 + 4sX(s) - 4 + 5X(s) = R(s)
\]

Collecting terms: \((s^2 + 4s + 5)X(s) = R(s) + s + 3\)

Solving for \(X(s)\),

\[
X(s) = \frac{R(s)}{s^2 + 4s + 5} + \frac{s + 3}{s^2 + 4s + 5}
\]

The block diagram is shown below, where \(R(s) = 1/s\).

13.

Program:

'Factored'
\[Gzpk=zpk([-15 -26 -72],[0 -55 roots([1 5 30])' roots([1 27 52])'],5)\]

'Polynomial'
\[Gp=tf(Gzpk)\]

Computer response:
\[\text{ans} = \]

\[
\begin{align*}
\frac{1}{s^2 + 4s + 5}
\end{align*}
\]

\[
\begin{align*}
R(s)
\end{align*}
\]

\[
\begin{align*}
+ \quad + \quad + \quad +
\end{align*}
\]

\[
\begin{align*}
X(s)
\end{align*}
\]
Factored

Zero/pole/gain:
\[ \frac{5(s+15)(s+26)(s+72)}{s(s+55)(s+24.91)(s+2.087)(s^2+5s+30)} \]

\[ \text{ans} = \]

Polynomial

Transfer function:
\[ \frac{5s^3 + 565s^2 + 16710s + 140400}{s^6 + 87s^5 + 1977s^4 + 1.301e004s^3 + 6.041e004s^2 + 8.58e004s} \]

14.

Program:

'Polynomial'

\[ \text{Gtf}=\text{tf}([1 \ 25 \ 20 \ 15 \ 42],[1 \ 13 \ 9 \ 37 \ 35 \ 50]) \]

'Factored'

\[ \text{Gzpk} = \text{zpk}(\text{Gtf}) \]

Computer response:

\[ \text{ans} = \]

Polynomial

Transfer function:
\[ \frac{s^4 + 25s^3 + 20s^2 + 15s + 42}{s^5 + 13s^4 + 9s^3 + 37s^2 + 35s + 50} \]

\[ \text{ans} = \]

Factored

Zero/pole/gain:
\[ \left( s+24.2 \right) \left( s+1.35 \right) \left( s^2 - 0.5462s + 1.286 \right) \]

\[ \frac{\left( s+12.5 \right) \left( s^2 + 1.463s + 1.493 \right) \left( s^2 - 0.964s + 2.679 \right)}{(s+2) + 1.463s + 1.493) (s^2 - 0.964s + 2.679)} \]

15.

Program:

\[ \text{numg}=[-5 -70]; \]
\[ \text{deng}=[0 -45 -55 \text{ (roots([1 7 110]))}' \text{ (roots([1 6 95]))}')]; \]
\[ \text{[numg,deng]} = \text{zp2tf(numg',deng',1e4)}; \]
\[ \text{Gtf}=\text{tf(numg,deng)} \]
\[ \text{G} = \text{zpk(Gtf)} \]
\[ \text{[r,p,k]} = \text{residue(numg,deng)} \]

Computer response:
Transfer function:
\[
\frac{10000 s^2 + 750000 s + 3.5 \times 10^6}{s^7 + 113 s^6 + 4022 s^5 + 58200 s^4 + 754275 s^3 + 4.324 \times 10^6 s^2 + 2.586 \times 10^7 s}
\]

Zero/pole/gain:
\[
\frac{10000 (s+70) (s+5)}{s (s+55) (s+45) (s^2 + 6s + 95) (s^2 + 7s + 110)}
\]

\[
r = \begin{cases} 
-0.0018 \\
0.0066 \\
0.9513 + 0.0896i \\
0.9513 - 0.0896i \\
-1.0213 - 0.1349i \\
-1.0213 + 0.1349i \\
0.1353 
\end{cases}
\]

\[
p = \begin{cases} 
-55.0000 \\
-45.0000 \\
-3.5000 + 9.8869i \\
-3.5000 - 9.8869i \\
-3.0000 + 9.2736i \\
-3.0000 - 9.2736i \\
0 
\end{cases}
\]

\[
k = \begin{bmatrix} [] \end{bmatrix}
\]

16. Program:
```matlab
syms s
Ga=45*[(s^2+37*s+74)*(s^3+28*s^2+32*s+16)]... /
[(s+39)*(s+47)*(s^2+2*s+100)*(s^3+27*s^2+18*s+15)];
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);
pretty(Ga)
Ga=tf(numga,denga)
Ga=zpk(Ga)
Ga=56*[(s+14)*(s^2+39*s+39)]... /
[(s^2+88*s+33)*(s^2+56*s+77)*(s^3+81*s^2+76*s+65)];
pretty(Ga)
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
```
denga=sym2poly(denga);
'Ga polynomial'
Ga=tf(numga,denga)
'Ga factored'
Ga=zpk(Ga)

**Computer response:**

(a)

ans =
Ga symbolic

\[
\frac{2}{(s^2 + 37s + 74)} \cdot \frac{3}{(s^2 + 28s + 32 + 16)}
\]
\[
\frac{45}{(s + 39)(s + 47)(s^2 + 2s + 100)(s^2 + 27s + 18s + 15)}
\]

ans =
Ga polynomial
Transfer function:

\[
45s^5 + 2925s^4 + 51390s^3 + 147240s^2 + 133200s + 53280
\]
\[
\frac{1}{s^7 + 115s^6 + 4499s^5 + 70700s^4 + 553692s^3 + 5.201e006s^2 + 3.483e006s + 2.75e006}
\]

ans =
Ga factored
Zero/pole/gain:

\[
45(s + 34.88)(s + 26.83)(s + 2.122)(s^2 + 1.17s + 0.5964)
\]
\[
\frac{1}{(s + 47)(s + 39)(s + 26.34)(s^2 + 0.6618s + 0.5695)(s^2 + 2s + 100)}
\]

(ans)

(b)

ans =
Ga symbolic

\[
\frac{3}{2}
\]
Chapter 2: Modeling in the Frequency Domain

\[
\frac{(s + 14)(s + 49s + 62s + 53)}{(s + 88s + 33)(s + 56s + 77)(s + 81s^2 + 76s + 65)}
\]

\[
\frac{56}{s^7 + 225s^6 + 16778s^5 + 427711s^4 + 1.093e006s^3 + 1.189e006s^2 + 753676s + 165165}
\]

\[
\frac{56(s + 47.72)(s + 14)(s^2 + 1.276s + 1.111)}{(s + 87.62)(s + 80.06)(s + 54.59)(s + 1.411)(s + 0.3766)(s^2 + 0.9391s + 0.8119)}
\]

17.

a. Writing the node equations, \( \frac{V_i - V_o}{s} + \frac{V_o}{s} + V_o = 0 \). Solve for \( \frac{V_o}{V_i} = \frac{1}{s+2} \).

b. Thevenizing,

Using voltage division, \( V_o(s) = \frac{V_o(t)}{2} \). Thus, \( \frac{V_o(s)}{V_i(s)} = \frac{1}{2s^2 + s + 2} \).
a.

Writing mesh equations

\[(2s + 2)I_1(s) - 2I_2(s) = V_i(s)\]

\[-2I_1(s) + (2s + 4)I_2(s) = 0\]

But from the second equation, \(I_1(s) = (s + 2)I_2(s)\). Substituting this in the first equation yields,

\[(2s + 2)(s + 2)I_2(s) - 2I_2(s) = V_i(s)\]

or

\[I_2(s)/V_i(s) = 1/(2s^2 + 4s + 2)\]

But, \(V_L(s) = sI_2(s)\). Therefore, \(V_L(s)/V_i(s) = s/(2s^2 + 4s + 2)\).

b.

\[(4 + \frac{4}{s})I_1(s) - (2 + \frac{2}{s})I_2(s) = V(s)\]

\[-(2 + \frac{2}{s})I_1(s) + (4 + \frac{2}{s} + 2s)I_2(s) = 0\]

Solving for \(I_2(s)\):
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Therefore, \[ V_i(s) \over V(s) = 2sI_1(s) \over V(s) = \frac{2s^2}{4s^2 + 6s + 2} = \frac{s^2}{2s^2 + 3s + 1} \]

19.

a. 

Writing mesh equations,

\[(2s + 1)I_1(s) - I_2(s) = V_i(s) \]

\[-I_1(s) + (3s + 1 + 2/s)I_2(s) = 0 \]

Solving for \( I_2(s) \),

\[ I_2(s) = \begin{bmatrix} 2s + 1 & V_i(s) \\ -1 & 0 \end{bmatrix} \]

Solving for \( I_2(s)/V_i(s) \),

\[ \frac{I_2(s)}{V_i(s)} = \begin{bmatrix} 2s + 1 & -1 \\ -1 & 3s^2 + s + 2 \end{bmatrix} \]

But \( V_o(s) = I_2(s)3s \). Therefore, \( G(s) = 3s^2/(6s^3 + 5s^2 + 4s + 2) \).

b. Transforming the network yields,
Writing the loop equations,

\[(s + \frac{s}{s^2 + 1}I_1(s) - \frac{s}{s^2 + 1}I_2(s) - sI_3(s) = V_i(s)\]

\[-\frac{s}{s^2 + 1}I_1(s) + (\frac{s}{s^2 + 1} + 1 + \frac{1}{s})I_2(s) - I_3(s) = 0\]

\[-sI_1(s) - I_2(s) + (2s + 1)I_3(s) = 0\]

Solving for \(I_2(s)\),

\[I_2(s) = \frac{s(s^2 + 2s + 2)}{s^4 + 2s^3 + 3s^2 + 3s + 2} \frac{V_i(s)}{}\]

But, \(V_o(s) = \frac{I_2(s)}{s} = \frac{s^2 + 2s + 2}{s^4 + 2s^3 + 3s^2 + 3s + 2} V_i(s)\). Therefore,

\[\frac{V_o(s)}{V_i(s)} = \frac{s^2 + 2s + 2}{s^4 + 2s^3 + 3s^2 + 3s + 2}\]

20. a. Writing the nodal equations yields,

\[\frac{V_R(s) - V_i(s)}{2s} + \frac{V_R(s)}{1} + \frac{V_R(s) - V_C(s)}{3s} = 0\]

\[-\frac{1}{3s} V_R(s) + \left(\frac{1}{2} + \frac{1}{3s}\right) V_C(s) = 0\]

Rewriting and simplifying,

\[\frac{6s + 5}{6s} \frac{V_R(s)}{s} - \frac{1}{3s} V_C(s) = \frac{1}{2s} V_i(s)\]

\[-\frac{1}{3s} V_R(s) + \left(\frac{3s^2 + 2}{6s}\right) V_C(s) = 0\]
Solving for $V_R(s)$ and $V_C(s)$,

$$V_R(s) = \begin{vmatrix} \frac{1}{2s} & 0 \\ \frac{1}{3s} & 3s^2 + 2 \end{vmatrix} \quad \text{and} \quad V_C(s) = \begin{vmatrix} \frac{6s + 5}{6s} & \frac{1}{2s} \\ \frac{3s^2 + 2}{6s} & 0 \end{vmatrix}$$

Solving for $V_o(s)/V_i(s)$

$$V_o(s) = \frac{V_R(s) - V_C(s)}{V_i(s)} = \frac{3s^2}{6s^4 + 5s^2 + 4s + 2}$$

b. Writing the nodal equations yields,

$$\frac{(V_i(s) - V_i(s))}{s} + \frac{(s^2 + 1)}{s} V_i(s) + (V_i(s) - V_o(s)) = 0$$

$$V_o(s) - V_i(s) + sV_o(s) + \frac{(V_o(s) - V_i(s))}{s} = 0$$

Rewriting and simplifying,

$$\left(s + \frac{2}{s} + 1\right)V_i(s) - V_o(s) = \frac{1}{s} V_i(s)$$

$$V_i(s) + (s + \frac{1}{s} + 1)V_o(s) = \frac{1}{s} V_i(s)$$

Solving for $V_o(s)$

$$V_o(s) = \frac{(s^2 + 2s + 2)}{s^4 + 2s^3 + 3s^2 + 3s + 2} V_i(s)$$

Hence,
$$V_v(s) = \frac{(s^2 + 2s + 2)}{J(s)}$$

21. 

a. 

Mesh:

\[(4+4s)I_1(s) - (2+4s)I_2(s) - 2I_3(s) = V(s)\]

\[-(2+4s)I_1(s) + (14+10s)I_2(s) - (4+6s)I_3(s) = 0\]

\[-2I_1(s) - (4+6s)I_2(s) + (6+6s+ \frac{9}{s})I_3(s) = 0\]

Nodal:

\[\frac{(V_v(s) - V(s))}{2} + \frac{V_i(s)}{2+4s} + \frac{(V_v(s) - V(s))}{4+6s} = 0\]

\[\frac{(V_v(s) - V_i(s))}{4+6s} + \frac{V_i(s)}{8} + \frac{(V_v(s) - V(s))}{9/s} = 0\]

or

\[\left[\frac{6s^2 + 12s + 5}{12s^2 + 14s + 4}\right]V_i(s) - \left[\frac{1}{6s+4}\right]V_v(s) = \frac{1}{2}V(s)\]

\[-\left[\frac{1}{6s+4}\right]V_i(s) + \left[\frac{24s^2 + 43s + 54}{216s+144}\right]V_v(s) = \frac{s}{9}V(s)\]

b. 

Program:

```matlab
syms s V
\%Construct symbolic object for frequency
```
Chapter 2: Modeling in the Frequency Domain

'Mesh Equations'
\[
A_2 = \begin{bmatrix}
(4+4s) & V & -2 \\
-(2+4s) & 0 & -(4+6s) \\
-2 & 0 & (6+6s+(9/s))
\end{bmatrix}
\]
%Form Ak = A2.

A = \begin{bmatrix}
(4+4s) & (2+4s) & -2 \\
-(2+4s) & (14+10s) & -(4+6s) \\
-2 & -(4+6s) & (6+6s+(9/s))
\end{bmatrix}
%Form A.

I_2 = \frac{\det(A_2)}{\det(A)};
%Use Cramer's Rule to solve for I_2.

G_i = I_2/V;
%Form transfer function, G_i(s) = I_2(s)/V(s).

G = 8*G_i;
%Form transfer function, G(s) = 8*I_2(s)/V(s).

G = collect(G);
%Simplify G(s).

'\text{G(s) via Mesh Equations}'
%Display label.

pretty(G)
%Pretty print G(s)

'Nodal Equations'
\[
A_2 = \begin{bmatrix}
(6s^2+12s+5)/(12s^2+14s+4) & V/2 \\
-1/(6s+4) & s*(V/9)
\end{bmatrix}
\]
%Form Ak = A2.

A = \begin{bmatrix}
(6s^2+12s+5)/(12s^2+14s+4) & -1/(6s+4) \\
-1/(6s+4) & (24s^2+43s+54)/(216s+144)
\end{bmatrix}
%Form A.

V_o = \frac{\text{simple}(\det(A_2))}{\text{simple}(\det(A))};
%Use Cramer's Rule to solve for V_o.

G_1 = V_o/V;
%Form transfer function, G_1(s) = V_o(s)/V(s).

G_1 = collect(G_1);
%Simplify G_1(s).

'\text{G(s) via Nodal Equations}'
%Display label.

pretty(G_1)
%Pretty print G_1(s)

Computer response:
ans =

Mesh Equations

\[ A_2 = \]
\[
\begin{bmatrix}
4s + 4, V, & -2 \\
-4s - 2, 0, & -6s - 4 \\
-2, 0, 6s + 9/s + 6
\end{bmatrix}
\]

\[ A = \]
\[
\begin{bmatrix}
4s + 4, -4s - 2, & -2 \\
-4s - 2, 10s + 14, & -6s - 4 \\
-2, -6s - 4, 6s + 9/s + 6
\end{bmatrix}
\]

ans =

\[ G(s) \text{ via Mesh Equations} \]
\[
\frac{3s^2 + 96s + 112s + 36}{3s^2 + 150s + 220s + 117}
\]

ans =

Nodal Equations

\[ A_2 = \]
\[
\begin{bmatrix}
(6s^2 + 12s + 5)/(12s^2 + 14s + 4), & V/2 \\
-1/(6s + 4), & (Vs)/9
\end{bmatrix}
\]

\[ A = \]
\[
\begin{bmatrix}
(6s^2 + 12s + 5)/(12s^2 + 14s + 4), & -1/(6s + 4) \\
-1/(6s + 4), & (24s^2 + 43s + 54)/(216s + 144)
\end{bmatrix}
\]

ans =

\[ G(s) \text{ via Nodal Equations} \]
\[
3s^2 + 12s + 5
\]
22.  

a.  

\[ Z_1(s) = 5 \times 10^5 + \frac{1}{2 \times 10^6 s} \]

\[ Z_2(s) = 10^5 + \frac{1}{2 \times 10^6 s} \]

Therefore,

\[ \frac{-Z_2(s)}{Z_1(s)} = -\frac{1}{5} \left( \frac{s+5}{s+1} \right) \]

b.  

\[ Z_1(s) = 10^5 \left( \frac{5}{s} + 1 \right) = 10^5 \left( \frac{s+5}{s} \right) \]

\[ Z_2(s) = 10^5 \left( 1 + \frac{5}{s+5} \right) = 10^5 \left( \frac{s+10}{s+5} \right) \]

Therefore,

\[ \frac{-Z_2(s)}{Z_1(s)} = -s \left( \frac{s+10}{s+5} \right)^2 \]

23.  

a.  

\[ Z_1(s) = 4 \times 10^5 + \frac{1}{4 \times 10^{-6} s} \]

\[ Z_2(s) = 1.1 \times 10^5 + \frac{1}{4 \times 10^{-6} s} \]

Therefore,

\[ G(s) = \frac{Z_2(s) + Z_2(s)}{Z_1(s)} = 1.275 \left( \frac{s+0.98}{s+0.625} \right) \]

b.  


Solutions to Problems

\[ Z_1(s) = 4 \times 10^3 + \frac{10^{11}}{s} \]  
\[ Z_2(s) = 6 \times 10^3 + \frac{27.5 \times 10^9}{s} \]  

Therefore,

\[ \frac{Z_1(s) + Z_2(s)}{Z_1(s)} = \frac{2640s^2 + 8420s + 4275}{1056s^2 + 3500s + 2500} \]

24. Writing the equations of motion, where \( x_2(t) \) is the displacement of the right member of spring,

\[ (5s^2 + 4s + 5)X_1(s) - 5X_2(s) = 0 \]  
\[ -5X_1(s) + 5X_2(s) = F(s) \]

Adding the equations,

\[ (5s^2 + 4s)X_1(s) = F(s) \]

From which,

\[ \frac{X_1(s)}{F(s)} = \frac{1}{s(5s + 4)} = \frac{1/5}{s(s + 4/5)}. \]

25. Writing the equations of motion,

\[ (s^2 + s + 1)X_1(s) - (s + 1)X_2(s) = F(s) \]  
\[ -(s + 1)X_1(s) + (s^2 + s + 1)X_2(s) = 0 \]

Solving for \( X_2(s) \),

\[ X_2(s) = \left[ \begin{array}{c} (s^2 + s + 1) & F(s) \\ -s-1 & 0 \end{array} \right] = \frac{(s + 1)F(s)}{s^2(s^2 + 2s + 2)} \]

From which,

\[ \frac{X_2(s)}{F(s)} = \frac{(s + 1)}{s^2(s^2 + 2s + 2)}. \]

26. Let \( X_1(s) \) be the displacement of the left member of the spring and \( X_3(s) \) be the displacement of the mass. Writing the equations of motion, gives:
\[ 2X_1(s) - 2X_2(s) = F(s) \]
\[ -2X_1(s) + (4s + 2)X_2(s) - 4sX_3(s) = 0 \]
\[ -4sX_2(s) + (8s^2 + 6s)X_3(s) = 0 \]

The third equation may be rewritten as: 
\[ -2X_2(s) + (4s + 3)X_3(s) = 0 \]

From which we get: 
\[ X_3(s) = \frac{2}{4s + 3} X_2(s) \]

Substituting for \( X_3(s) \) into the second equation and simplifying, gives the following set of two equations:
\[ 2X_1(s) - 2X_2(s) = F(s) \]
\[ -(4s + 3)X_1(s) + (8s^2 + 6s + 3)X_2(s) = 0 \]

Solving for \( X_2(s) \),
\[
X_2(s) = \frac{2F(x)}{-2(4s + 3)F(s) - 2(8s^2 + 6s + 3)}
\]
\[ \frac{X_2(s)}{F(s)} = \frac{4s + 3}{8s(2s + 1/2)} \]

Thus,
\[ G(s) = \frac{X_1(s)}{F(s)} = \frac{4s + 3}{2s^4 + 17s^3 + 44s^2 + 45s + 20} \]

Writing the equations of motion,
Solutions to Problems 2-25

(4s^2 + 2s + 6)X_1(s) - 2sX_2(s) = 0
-2sX_1(s) + (4s^2 + 4s + 6)X_2(s) - 6X_3(s) = F(s)
-6X_2(s) + (4s^2 + 2s + 6)X_3(s) = 0

Solving for $X_3(s)$,

$$X_3(s) = \frac{(4s^2 + 2s + 6) - 2s}{(4s^2 + 2s + 6) - 2s - 2s(4s^2 + 4s + 6) - 6} = \frac{3F(s)}{s(8s^3 + 12s^2 + 26s + 18)}$$

From which,

$$\frac{X_3(s)}{F(s)} = \frac{3}{s(8s^3 + 12s^2 + 26s + 18)}.$$

29.

a.

(4s^2 + 8s + 5)X_1(s) - 8sX_2(s) - 5X_3(s) = F(s)
-8sX_1(s) + (4s^2 + 16s)X_2(s) - 4sX_3(s) = 0
-5X_1(s) - 4sX_2(s) + (4s + 5)X_3(s) = 0

Solving for $X_3(s)$,

$$X_3(s) = \frac{(4s^2 + 8s + 5) - 8s}{(4s^2 + 8s + 5) - 8s - 8s(4s^2 + 16s) - 0} = \frac{F(s)}{\Delta}$$

or,

$$\frac{X_3(s)}{F(s)} = \frac{13s + 20}{4s(4s^3 + 25s^2 + 43s + 15)}$$

b.
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\[
(8s^2 + 4s + 16)X_1(s) - (4s + 1)X_2(s) - 15X_3(s) = 0 \\
-(4s + 1)X_1(s) + (3s^2 + 20s + 1)X_2(s) - 16sX_3(s) = F(s) \\
-15X_1(s) - 16sX_2(s) + (16s + 15)X_3(s) = 0
\]

Solving for \(X_3(s)\),

\[
X_1(s) = \frac{\begin{vmatrix} (8s^2 + 4s + 16) & -(4s+1) & 0 \\ -(4s+1) & (3s^2 + 20s + 1) & F(s) \\ -15 & -16s & 0 \end{vmatrix}}{\Delta} = \frac{-(8s^2 + 4s + 16) - (4s+1)}{-15 - 16s} \\
\]

or

\[
\frac{X_3(s)}{F(s)} = \frac{128s^3 + 64s^2 + 316s + 15}{384s^5 + 1064s^4 + 3476s^3 + 165s^2}
\]

30. Writing the equations of motion,

\[
(4s^2 + 4s + 8)X_1(s) - 4X_2(s) - 2sX_3(s) = 0 \\
-4X_1(s) + (5s^2 + 3s + 4)X_2(s) - 3sX_3(s) = F(s) \\
-2sX_1(s) - 3sX_2(s) + (5s^2 + 5s + 5) = 0
\]

31. Using the impedance method the two equations are:

\[
x_1: \quad (ms^2 + k)x_1 - x_mk = F_1 \\
x_m: \quad -x_1k + (Bs + k)x_m = F_{iso}
\]

Solving both equations simultaneously, one gets

\[
x_1 = \begin{vmatrix} F_1 & -k \\ F_{iso} & Bs + k \end{vmatrix} = \frac{F_1(Bs + k) + F_{iso}k}{(ms^2 + k)(Bs + k) - k^2} = \frac{F_1Bs + k(F_1 + F_{iso})}{s(mBs^2 + kms + kB)}
\]

32.
a.

Writing the equations of motion,

\[ (5s^2 + 9s + 9)\theta_1(s) - (s + 9)\theta_2(s) = 0 \]
\[ -(s + 9)\theta_1(s) + (3s^2 + s + 12)\theta_2(s) = T(s) \]

b.

Defining
\[ \theta_1(s) = \text{rotation of } J_1 \]
\[ \theta_2(s) = \text{rotation between } K_1 \text{ and } D_1 \]
\[ \theta_3(s) = \text{rotation of } J_3 \]
\[ \theta_4(s) = \text{rotation of right-hand side of } K_2 \]

the equations of motion are

\[ (J_1s^2 + K_1)\theta_1(s) - K_1\theta_2(s) = T(s) \]
\[ -K_1\theta_1(s) + (D_1s + K_3)\theta_3(s) - D_3\theta_4(s) = 0 \]
\[ -D_3\theta_3(s) + (J_1s^2 + D_3s + K_2)\theta_3(s) - K_2\theta_2(s) = 0 \]
\[ -K_2\theta_3(s) + (D_3s + (K_2 + K_1))\theta_4(s) = 0 \]

33.

Writing the equations of motion,

\[ (s^2 + 2s + 1)\theta_1(s) - (s + 1)\theta_2(s) = T(s) \]
\[ -(s + 1)\theta_1(s) + (2s + 1)\theta_2(s) = 0 \]

Solving for \( \theta_2(s) \)

\[ \theta_2(s) = \frac{\begin{bmatrix} (s^2 + 2s + 1) & T(s) \\ -(s + 1) & 0 \end{bmatrix}}{\begin{bmatrix} (s^2 + 2s + 1) & -(s + 1) \\ -(s + 1) & (2s + 1) \end{bmatrix}} = \frac{T(s)}{2s(s + 1)} \]

Hence,

\[ \frac{\theta_2(s)}{T(s)} = \frac{1}{2s(s + 1)} \]

34.

The corresponding impedance equations are:
\[ \theta_1 : \quad (s^2 + s + 1)\theta_1 - (s + 1)\theta_2 = T \]
\[ \theta_2 : \quad -(s+1)\theta_1 + (s^2 + s + 2)\theta_2 = 0 \]

Solving for \( \theta_1 \) one gets:
\[ \theta_1 = \frac{\begin{vmatrix} T & -(s+1) \\ s^2 + s + 2 & 0 \end{vmatrix}}{\begin{vmatrix} s^2 + s + 1 & -(s+1) \\ -(s+1) & s^2 + s + 2 \end{vmatrix}} = \frac{T(s^2 + s + 2)}{(s^2 + s + 1)(s^2 + s + 2) - (s+1)^2} \]

Simplifying:
\[ \frac{\theta_1}{T} = \frac{s^2 + s + 2}{s^4 + 2s^3 + 3s^2 + 1} \]

35.
Reflecting impedances to \( \theta_3 \),
\[ (J_{eq}s^2 + D_{eq}s)\theta_3(s) = T(s) \left( \frac{N_4N_2}{N_3N_1} \right) \]

Thus,
\[ \frac{\theta_2(s)}{T(s)} = \frac{N_4N_2}{N_3N_1} \frac{1}{J_{eq}s^2 + D_{eq}s} \]

where
\[ J_{eq} = J_4 + J_5 + (J_2 + J_3) \left( \frac{N_4}{N_3} \right)^2 + J_1 \left( \frac{N_4N_2}{N_3N_1} \right)^2, \text{ and} \]
\[ D_{eq} = (D_4 + D_5) + (D_2 + D_3) \left( \frac{N_4}{N_3} \right)^2 + D_1 \left( \frac{N_4N_2}{N_3N_1} \right)^2 \]

36.
Reflecting all impedances to \( \theta_2(s) \),
\[ \left\{ \left[ J_2 + J_1 \left( \frac{N_4}{N_3} \right)^2 + J_3 \left( \frac{N_4}{N_4} \right)^2 \right]s^2 + \left[ f_2 + f_1 \left( \frac{N_4}{N_4} \right)^2 + f_3 \left( \frac{N_3}{N_4} \right)^2 \right]s + \left[ K \left( \frac{N_4}{N_4} \right)^2 \right] \right\} \theta_2(s) = T(s) \frac{N_2}{N_1} \]

Substituting values,
\[ \left\{ \left[ 1 + 2(3)^2 + 16 \left( \frac{1}{4} \right)^2 \right]s^2 + \left[ 2 + 1(3)^2 + 32 \left( \frac{1}{4} \right)^2 \right]s + 64 \left( \frac{1}{4} \right)^2 \right\} \theta_2(s) = T(s)(3) \]
Thus,
\[
\frac{\theta_3(s)}{T(s)} = \frac{3}{20s^2 + 13s + 4}
\]

37.
Reflecting impedances across gears from the right hand side to the left hand side one gets:

\[
J_{eq} = 3 + 100\left(\frac{5}{25}\right)^2 + 150\left(\frac{5}{20}\right)^2 = 9
\]

\[
D_{eq} = 500\left(\frac{5}{25}\right)^2 + 300\left(\frac{5}{50}\right)^2 = 23
\]

\[
K_{eq} = 3 + 300\left(\frac{5}{50}\right)^2 = 6
\]

So \((9s^2 + 23s + 6)\theta(s) = T(s)\). Since \(\frac{\theta}{\theta_2} = \frac{N_s}{N_1} = 10\), \((9s^2 + 23s + 6)10\theta_2(s) = T(s)\)

\[
\frac{\theta_2(s)}{T(s)} = \frac{1}{90s^2 + 230s + 60} = \frac{0.011}{s^2 + 2.55s + 0.67} = \frac{0.011}{(s + 0.29)(s + 2.26)}
\]

38.
Reflecting impedances and applied torque to respective sides of the spring yields the following equivalent circuit:

Writing the equations of motion,
\[
2\theta_2(s) - 2\theta_3(s) = 4.231T(s)
\]
\[
-2\theta_2(s) + (0.955s + 2)\theta_3(s) = 0
\]

Solving for \(\theta_3(s)\),
\[
\theta_3(s) = \frac{1}{2}
\begin{bmatrix}
2 & 4.231T(s) \\
-2 & 0 \\
-2 & 1.91s
\end{bmatrix}
= \frac{8.462T(s)}{1.91s} = \frac{4.43T(s)}{s}
\]

Hence, \(\frac{\theta_3(s)}{T(s)} = \frac{4.43}{s}\). But, \(\theta_3(s) = 0.192\theta_1(s)\). Thus, \(\frac{\theta_1(s)}{T(s)} = \frac{0.851}{s}\).
Reflecting the 0.02 Nm/rad damper towards the left we get

\[
\begin{align*}
\theta_1 : \quad & (s^2 + 2s)\theta_1 - 2s\theta_2 = T_1 \\
\theta_2 : \quad & -2s\theta_1 + (2.32s + 2)\theta_2 = 0 
\end{align*}
\]

Solving:

\[
\theta_2 = \frac{(s^2 + 2s)\theta_1}{(s^2 + 2s)(2.32s + 2) - 4s^2}
\]

\[
= \frac{2sT_1}{2.32s^3 + 2s^2 + 4.64s^2 + 4s - 4s^2} = \frac{2T_1}{2.32s^2 + 2.64s + 4}
\]

So

\[
\frac{\theta_2}{T_1} = \frac{2}{2.32s^2 + 2.64s + 4}
\]

Using the gear ratios we get

\[
\frac{T}{T_1} = \frac{5}{20} = \frac{1}{4} \quad \text{and} \quad \frac{\theta_2}{\theta_L} = \frac{10}{40} = \frac{1}{4}. \quad \text{It follows that} \quad \frac{\theta_2}{T_1} = \frac{\theta_2}{4T} = \frac{1}{16} \frac{\theta_L}{T}
\]

Finally

\[
\frac{\theta_2}{T} = \frac{32}{2.32s^2 + 2.64s + 4} = \frac{13.8}{s^2 + 1.14s + 1.72}
\]

Reflect all impedances on the right to the viscous damper and reflect all impedances and torques on the left to the spring and obtain the following equivalent circuit:
Writing the equations of motion,

\[(J_{1eq}^{2} + K)\dot{\theta}_{2}(s) - K\dot{\theta}_{3}(s) = T_{eq}(s)\]
\[-K\dot{\theta}_{2}(s) + (D_{eq} + K)\dot{\theta}_{3}(s) - D_{eq}\dot{\theta}_{4}(s) = 0\]
\[-D_{eq}\dot{\theta}_{3}(s) + [J_{2eq}^{2} + (D_{eq} + D_{eq})s]\dot{\theta}_{4}(s) = 0\]

where: \(J_{1eq} = J_{2} + (J_{a} + J_{1}) \left( \frac{N_{2}}{N_{1}} \right)^{2} ; J_{2eq} = J_{3} + (J_{L} + J_{4}) \left( \frac{N_{4}}{N_{4}} \right)^{2} ; D_{eq} = D_{L} \left( \frac{N_{4}}{N_{4}} \right)^{2} ; \theta_{2}(s) = \theta_{1}(s)\]

\(\frac{N_{1}}{N_{2}}\).

41.

Reflect impedances to the left of J₅ to J₅ and obtain the following equivalent circuit:

Writing the equations of motion,

\[[J_{eq}^{2} + (D_{eq} + D)s + (K_{2} + K_{eq})]\dot{\theta}_{5}(s) - [D_{eq} + K_{2}]\dot{\theta}_{6}(s) = 0\]
\[-[K_{2} + D_{eq}]\dot{\theta}_{5}(s) + [J_{6eq}^{2} + 2D_{eq} + K_{2}]\dot{\theta}_{6}(s) = T(s)\]

From the first equation, \(\frac{\dot{\theta}_{6}(s)}{\dot{\theta}_{5}(s)} = \frac{J_{eq}^{2} + (D_{eq} + D)s + (K_{2} + K_{eq})}{D_{eq} + K_{2}}\). But, \(\frac{\dot{\theta}_{6}(s)}{\dot{\theta}_{5}(s)} = \frac{N_{1}N_{3}}{N_{2}N_{4}}\).

Therefore,

\(\frac{\dot{\theta}_{6}(s)}{\dot{\theta}_{5}(s)} = \frac{N_{1}N_{3}}{N_{2}N_{4}} \left( \frac{J_{eq}^{2} + (D_{eq} + D)s + (K_{2} + K_{eq})}{D_{eq} + K_{2}} \right)\),

where \(J_{eq} = J_{1} \left( \frac{N_{4}N_{2}}{N_{3}N_{4}} \right)^{2} + (J_{2} + J_{3}) \left( \frac{N_{4}}{N_{3}} \right)^{2} + (J_{4} + J_{5})\), \(K_{eq} = K_{1} \left( \frac{N_{4}}{N_{3}} \right)^{2}\), and
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\[ D_{eq} = D \left[ \left( \frac{N_2}{N_1} \right)^2 + \left( \frac{N_4}{N_3} \right)^2 + 1 \right]. \]

42. Draw the freebody diagrams,

Write the equations of motion from the translational and rotational freebody diagrams,

\[ (M_s^2 + 2f_v s + K_2)X(s) - f_v s \theta(s) = F(s) \]
\[ -f_v r s X(s) + (J_s^2 + f_v r^2 s) \theta(s) = 0 \]

Solve for \( \theta(s) \),

\[ \theta(s) = \frac{f_v r F(s)}{J M_s^3 + (2J f_v + M f_v r^2) s^2 + (JK_2 + f_v^2 r^2) s + K_2 f_v r^2} \]

From which,

\[ \frac{\theta(s)}{F(s)} = \frac{f_v r}{J M_s^3 + (2J f_v + M f_v r^2) s^2 + (JK_2 + f_v^2 r^2) s + K_2 f_v r^2}. \]

43. Draw a freebody diagram of the translational system and the rotating member connected to the translational system.
From the freebody diagram of the mass, \( F(s) = (2s^2 + 2s + 3)X(s) \). Summing torques on the rotating member,

\[
\begin{align*}
(J_{eq} & s^2 + D_{eq}s)\theta(s) + F(s)2 = T_{eq}(s) \quad \text{Substituting } F(s) \text{ above, (} J_{eq} s^2 + D_{eq}s)\theta(s) + (4s^2 + 4s + 6)X(s) = T_{eq}(s). \quad \text{However, } \theta(s) = \frac{X(s)}{2} \quad \text{Substituting and simplifying,} \\
T_{eq} &= \left[ \left( \frac{J_{eq}}{2} \right)s^2 + \left( \frac{D_{eq}}{2} \right)s + 4 \right]X(s) \\
\text{But, } J_{eq} = 3 + 3(4)^2 = 51, \quad D_{eq} = 1(2)^2 + 1 = 5, \quad \text{and } T_{eq}(s) = 4T(s). \quad \text{Therefore,} \\
\left[ \frac{59}{2} s^2 + \frac{13}{2} s + 6 \right]X(s) &= 4T(s). \quad \text{Finally, } \frac{X(s)}{T(s)} = \frac{8}{59s^2 + 13s + 12}.
\end{align*}
\]

44.

Writing the equations of motion,

\[
\begin{align*}
(J_1s^2 + K_1)\theta_1(s) - K_1\theta_2(s) &= T(s) \\
-K_1\theta_1(s) + (J_2s^2 + D_3s + K_1)\theta_2(s) + F(s)r - D_3\theta_3(s) &= 0 \\
-D_3\theta_2(s) + (J_2s^2 + D_3s)\theta_3(s) &= 0
\end{align*}
\]

where \( F(s) \) is the opposing force on \( J_2 \) due to the translational member and \( r \) is the radius of \( J_2 \). But, for the translational member,

\[
F(s) = (Ms^2 + f_v s + K_2)X(s) = (Ms^2 + f_v s + K_2)r\theta(s)
\]

Substituting \( F(s) \) back into the second equation of motion,

\[
\begin{align*}
(J_1s^2 + K_1)\theta_1(s) - K_1\theta_2(s) &= T(s) \\
-K_1\theta_1(s) + [(J_2 + Mr^2)s^2 + (D_3 + f_v r^2)s + (K_1 + K_2 r^2)]\theta_2(s) - D_3\theta_3(s) &= 0 \\
-D_3\theta_2(s) + (J_2s^2 + D_3s)\theta_3(s) &= 0
\end{align*}
\]

Notice that the translational components were reflected as equivalent rotational components by the
square of the radius. Solving for $\theta_2(s)$, 
$$\theta_2(s) = \frac{K_1(J_s s^2 + D_s s)T(s)}{\Delta},$$
where $\Delta$ is the determinant formed from the coefficients of the three equations of motion. Hence,

$$\frac{\theta_2(s)}{T(s)} = \frac{K_1(J_s s^2 + D_s s)}{\Delta}$$

Since

$$X(s) = r\theta_2(s), \quad \frac{X(s)}{T(s)} = \frac{rK_1(J_s s^2 + D_s s)}{\Delta}$$

45.
Reflecting through gears the inertia and damping from the load side to motor shaft one gets,

$$J_m = 4 + 36\left(\frac{50}{150}\right)^2 = 8 \quad \text{and} \quad D_m = 50 + 36\left(\frac{50}{150}\right)^2 = 54$$

Note from the motor load curve that $K_T = \frac{T_{\text{stall}}}{e_a} = \frac{150}{50} = 3$ and $K_b = \frac{e_a}{\omega_{\text{no-load}}} = \frac{50}{100} = \frac{1}{2}$.

Substituting all of the above, one gets

$$\frac{\theta_m}{E_a} = \frac{K_i}{R_a J_m} \frac{1}{s + \frac{1}{J_m}\left(D_m + \frac{K_K_b}{R_a}\right)} = \frac{0.375}{s(s + 7.1875)}$$

Noting that $\frac{\theta_m}{\theta_L} = \frac{N_2}{N_1} = 3$

$$\frac{\theta_L}{E_a} = \frac{0.125}{s(s + 7.1875)}$$

46.
The parameters are:

$$\frac{K_i}{R_a} = \frac{T_s}{E_a} = \frac{5}{5} = 1; \quad K_b = \frac{E_a}{\omega} = \frac{5}{\frac{600}{2\pi} \frac{1}{60}} = \frac{1}{4}$$

$$J_m = 18\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{2}\right)^2 + 1 = 3.125; \quad D_m = 36\left(\frac{1}{4}\right)^2 = 2.25$$

Thus,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{1}{\frac{3.125}{s(s + \frac{1}{3.125}(2.25 + (1)(\frac{1}{4}))}} = \frac{0.32}{s(s + 0.8)}$$
Solutions to Problems 2-35

Since: \( \theta_a(s) = \frac{1}{4} \theta_m(s) \); then:

\[
\frac{\theta_a(s)}{E_a(s)} = \frac{0.08}{s(s + 0.8)}
\]

47. The following torque-speed curve can be drawn from the data given:

![Torque-speed curve](image)

Therefore, \( \frac{K_a}{R_a} = \frac{T_{stat}}{E_a} = \frac{100}{12} \); \( K_b = \frac{E_a}{\omega_{no-load}} = \frac{12}{1333.33} \). Also, \( J_m = 7 + 105 \left( \frac{1}{6} \right)^2 = 9.92 \); \( D_m = \)

3. Thus,

\[
\frac{\theta_m(s)}{E_a(s)} = \frac{\left( \frac{100}{12} \right) 9.92}{s(s + \frac{1}{9.92}(3.075))} = \frac{0.84}{s(s + 0.31)}.
\]

Since \( \theta_L(s) = \frac{1}{6} \theta_m(s), \quad \theta_t(s) = \frac{0.14}{E_a(s)} = \frac{0.14}{s(s + 0.31)} \).

48. From Eqs. (2.45) and (2.46),

\[
R_a I_d(s) + K_p s \theta(s) = E_d(s) \tag{1}
\]

Also,

\[ T_m(s) = K_I I_d(s) = (J_m s^2 + D_m s) \theta(s) \]. Solving for \( \theta(s) \) and substituting into Eq. (1), and simplifying yields

\[
\frac{I_d(s)}{E_a(s)} = \frac{1}{R_a s + \frac{R_a D_m + K_b K_I}{R_d J_m}} \tag{2}
\]

Using \( T_m(s) = K_I I_d(s) \) in Eq. (2).
\[ T_m(s) = K_r \frac{s + \frac{D_m}{J_m}}{R_a s + \frac{R_a D_m + K_b K_r}{R_a J_m}} \]

49. For the rotating load, assuming all inertia and damping has been reflected to the load,
\((J_{eqL} s^2 + D_{eqL} s) \theta_L(s) + F(s) r = T_{eq}(s)\), where \(F(s)\) is the force from the translational system, \(r = 2\) is the radius of the rotational member, \(J_{eqL}\) is the equivalent inertia at the load of the rotational load and the armature, and \(D_{eqL}\) is the equivalent damping at the load of the rotational load and the armature.

Since \(J_{eqL} = 1(2)^2 + 1 = 5\), and \(D_{eqL} = 1(2)^2 + 1 = 5\), the equation of motion becomes, \((5s^2 + 5s) \theta_L(s) + F(s) r = T_{eq}(s)\). For the translational system, \((s^2 + s) X(s) = F(s)\). Since \(X(s) = 2 \theta_L(s)\), \(F(s) = (s^2 + s) 2 \theta_L(s)\). Substituting \(F(s)\) into the rotational equation, \((9s^2 + 9s) \theta_L(s) = T_{eq}(s)\). Thus, the equivalent inertia at the load is 9, and the equivalent damping at the load is 9. Reflecting these back to the armature, yields an equivalent inertia of \(\frac{9}{4}\) and an equivalent damping of \(\frac{9}{4}\). Finally, \(K_b = 1\). Hence, \(\theta_L(s) = \frac{4}{9} \frac{\theta_m(s)}{E_a(s)}\). Since \(\theta_L(s) = \frac{1}{2} \theta_m(s), \theta_L(s) = \frac{2}{9} \frac{s(s + \frac{13}{9})}{s(s + \frac{9}{4})}\). But \(X(s) = r \theta_L(s) = 2 \theta_L(s)\). therefore, \(X(s) = \frac{4}{9} \frac{s(s + \frac{13}{9})}{s(s + \frac{9}{4})}\).

50. The equations of motion in terms of velocity are:

\[
\begin{align*}
&\left[ M_1 s + (f_{v1} + f_{v3}) + \frac{K_1}{s} + \frac{K_2}{s} \right] V_1(s) - \frac{K_2}{s} V_2(s) - f_{v3} V_3(s) = 0 \\
&- \frac{K_2}{s} V_1(s) + \left[ M_3 s + (f_{v2} + f_{v4}) + \frac{K_2}{s} \right] V_2(s) - f_{v4} V_3(s) = F(s) \\
&- f_{v3} V_1(s) - f_{v4} V_2(s) + \left[ M_3 s + f_{v3} + f_{v4} \right] V_3(s) = 0
\end{align*}
\]

For the series analogy, treating the equations of motion as mesh equations yields...
In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields

\begin{align*}
(J_1 s + D_1 + \frac{K_1}{s}) \Omega_1 (s) - (D_1 + \frac{K_1}{s}) \Omega_2 (s) &= T (s) \\
-(D_1 + \frac{K_1}{s}) \Omega_1 (s) + (J_2 s + D_1 + \frac{(K_1 + K_2)}{s}) \Omega_2 (s) &= 0 \\
-\frac{K_2}{s} \Omega_2 (s) - D_2 \Omega_1 (s) + (D_2 + \frac{K_2}{s}) \Omega_4 (s) &= 0 \\
(J_3 s + D_2 + \frac{K_2}{s}) \Omega_3 (s) - D_3 \Omega_4 (s) &= 0
\end{align*}

For the series analogy, treating the equations of motion as mesh equations yields
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In the circuit, resistors are in ohms, capacitors are in farads, and inductors are in henries.

For the parallel analogy, treating the equations of motion as nodal equations yields

52.

An input $r_1$ yields $c_1 = 5r_1 + 7$. An input $r_2$ yields $c_2 = 5r_2 + 7$. An input $r_1 + r_2$ yields, $5(r_1 + r_2) + 7 = 5r_1 + 7 + 5r_2 = c_1 + c_2 - 7$. Therefore, not additive. What about homogeneity? An input of $Kr_1$ yields $c = 5Kr_1 + 7 ≠ Kc_1$. Therefore, not homogeneous. The system is not linear.

53. 

a. Let $x = \delta x + 0$. Therefore,

$$\ddot{\delta x} + 3 \dot{\delta x} + 2 \delta x = \sin (0 + \delta x)$$

But, $\sin (0 + \delta x) = \sin 0 + \frac{d \sin x}{dx} \bigg|_{x=0} \delta x = 0 + \cos x \bigg|_{x=0} \delta x = \delta x$

Therefore, $\ddot{\delta x} + 3 \dot{\delta x} + 2 \delta x = \delta x$. Collecting terms, $\ddot{\delta x} + 3 \dot{\delta x} + \delta x = 0$

b. Let $x = \delta x + \pi$. Therefore,
\[\ddot{x} + 3\dot{x} + 2\delta x = \sin(\pi + \delta x)\]

But, \(\sin(\pi + \delta x) = \sin \pi + \frac{d\sin x}{dx}\bigg|_{x=\pi} \delta x = 0 + \cos x \bigg|_{x=\pi} \delta x = -\delta x\).

Therefore, \(\ddot{x} + 3\dot{x} + 2\delta x = -\delta x\). Collecting terms, \(\ddot{x} + 3\dot{x} + 3\delta x = 0\).

54.

The truncated Taylor series expansion of \(f(x) = 3e^{-5x} \approx f(0) + f'(0)x = 3 - 15x\)

Letting \(x = \delta x\) and substituting for \(f(x)\) one gets

\[
\frac{d^3\delta x}{dt^3} + 10\frac{d^2\delta x}{dt^2} + 20\frac{d\delta x}{dt} + 15\delta x = 3 - 15\delta x.
\]

Simplifying

\[
\frac{d^3\delta x}{dt^3} + 10\frac{d^2\delta x}{dt^2} + 20\frac{d\delta x}{dt} + 30\delta x = 3.
\]

55.

The given curve can be described as follows:

\(f(x) = -6; -\infty < x < -3;\)

\(f(x) = 2x; -3 < x < 3;\)

\(f(x) = 6; 3 < x < +\infty\)

Thus,

a. \(\ddot{x} + 17\dot{x} + 50x = -6\)

b. \(\ddot{x} + 17\dot{x} + 50x = 2x\) or \(\ddot{x} + 17\dot{x} + 48x = 0\)

c. \(\ddot{x} + 17\dot{x} + 50x = 6\)

56.

The relationship between the nonlinear spring’s displacement, \(x_r(t)\) and its force, \(f_r(t)\) is

\[x_r(t) = 1 - e^{-f_r(t)}\]

Solving for the force, \(f_r(t) = -\ln(1 - x_r(t))\) \hspace{1cm} (1)

Writing the differential equation for the system by summing forces,

\[\frac{2d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} - \ln(1 - x(t)) = f(t)\] \hspace{1cm} (2)

Letting \(x(t) = x_0 + \delta x\) and \(f(t) = 1 + \delta f\), linearize \(\ln(1 - x(t))\).
\[
\ln(1-x) - \ln(1-x_0) = \left. \frac{d \ln(1-x)}{dx} \right|_{x=x_0} \delta x
\]

Solving for \( \ln(1-x) \),

\[
\ln(1-x) = \ln(1-x_0) - \left. \frac{1}{1-x} \right|_{x=x_0} \delta x = \ln(1-x_0) - \frac{1}{1-x_0} \delta x \quad (3)
\]

When \( f = 1 \), \( \delta x = 0 \). Thus from Eq. (1), \( 1 = -\ln(1-x_0) \).

Solving for \( x_0 \), \( 1 - x_0 = e^{-1} \), or \( x_0 = 0.6321 \).

Substituting \( x_0 = 0.6321 \) into Eq. (3),

\[
\ln(1-x) = \ln(1-0.6321) - \frac{1}{1-0.6321} \delta x = -1 - 2.718 \delta x
\]

Placing this value into Eq. (2) along with \( x(t) = x_0 + \delta x \) and \( f(t) = 1 + \delta f \), yields the linearized differential equation,

\[
2 \frac{d^2 \delta x}{dt^2} + 2 \frac{d \delta x}{dt} + 1 + 2.718 \delta x = 1 + \delta f
\]

or

\[
2 \frac{d^2 \delta x}{dt^2} + 2 \frac{d \delta x}{dt} + 2.718 \delta x = \delta f
\]

Taking the Laplace transform and rearranging yields the transfer function,

\[
\frac{\delta x(s)}{\delta f(s)} = \frac{1}{2s^2 + 2s + 2.718}
\]

57. The three equations are transformed into the Laplace domain:

\[
SS - S_0 = k_\psi \tilde{K}_S C - k_\psi S
\]

\[
CS = k_\psi (S - \tilde{K}_M C)
\]

\[
Ps = k_r C
\]

The three equations are algebraically manipulated to give:
\[ S = \frac{S_0}{s + k_p} + \frac{k_p \tilde{K}_S}{s + k_p} C \]

\[ C = \frac{Sk_p}{s + k_p \tilde{K}_M} \]

\[ P = \frac{k_p C}{s} \]

By direct substitutions it is obtained that:

\[ S = \frac{(s + k_p \tilde{K}_M)}{s^2 + k_p (1 + \tilde{K}_M)s + k_p^2 (\tilde{K}_M - \tilde{K}_S) + 20} S_0 \]

\[ C = \frac{k_p}{(s^2 + k_p (1 + \tilde{K}_M)s + k_p^2 (\tilde{K}_M - \tilde{K}_S))} S_0 \]

\[ P = \frac{k_p C}{s(s^2 + k_p (1 + \tilde{K}_M)s + k_p^2 (\tilde{K}_M - \tilde{K}_S))} S_0 \]

b.

\[ S(\infty) = \lim_{s \to 0} sS(s) = 0 \]

\[ C(\infty) = \lim_{s \to 0} sC(s) = 0 \]

\[ P(\infty) = \lim_{s \to 0} sP(s) = \frac{k_p C}{s} \]

\[ = \frac{k_p^2 S_0}{k_p (\tilde{K}_M - \tilde{K}_S) + k_p^2 (\tilde{K}_S - \tilde{K}_M)} \]

\[ = S_0 \]

58.

Eliminate \( T_{jat} \) by direct substitution. This results in

\[ J \frac{d^2 \theta}{dt^2} = -kJ \dot{\theta}(t) - \eta J \dot{\theta}(t) - \rho J \int_0^t \dot{\theta}(t) dt + T_d(t) \]

Obtaining Laplace transform on both sides of this equation and eliminating terms one gets that:

\[ \frac{\Theta}{T_d} (s) = \frac{1}{J^2} \frac{1}{s^2 + \eta s^2 - k s + \rho} \]
a.

We have that

\[ m_L \ddot{x}_{La} = m_L g \phi \]

\[ x_{La} = x_T - x_L \]

\[ x_L = L \phi \]

From the second equation

\[ \ddot{x}_{La} = \ddot{x}_T - \ddot{x}_L = \dot{v}_T - \dot{v}_L = g \phi \]

Obtaining Laplace transforms on both sides of the previous equation

\[ sV_T - Ls \Phi = g \Phi \quad \text{from which} \quad sV_T = \Phi(g + Ls^2) \]

so that

\[ \frac{\Phi}{V_T}(s) = \frac{s}{g + Ls^2} = \frac{1}{L} \frac{s}{s^2 + \frac{g}{L}} = \frac{1}{L} \frac{s}{s^2 + \omega_0^2} \]

b. Under constant velocity \( V_T(s) = \frac{V_0}{s} \) so the angle is

\[ \Phi(s) = \frac{V_0}{L} \frac{1}{s^2 + \omega_0^2} \]

Obtaining inverse Laplace transform

\[ \phi(t) = \frac{V_0}{L \omega_0} \sin(\omega_0 t) \], the load will sway with a frequency \( \omega_0 \).

c. From \( m_T \ddot{x}_T - f_T - m_L g \phi \) and Laplace transformation we get

\[ m_T s^2 X_T(s) = F_T - m_L g \Phi(s) = F_T - m_L g \frac{1}{L} \frac{s}{s^2 + \omega_0^2} V_T = F_T - m_L g \frac{1}{L} \frac{s}{s^2 + \omega_0^2} X_T \]

From which
\[
\frac{X_T}{F_T} = \frac{1}{s^2(m_T + \frac{m_Lg}{L} \frac{1}{s^2 + \omega_0^2})} = \frac{s^2 + \omega_0^2}{s^2(m_T(s^2 + \omega_0^2) + m_L\omega_0^2)} = \frac{1}{m_T} \frac{s^2 + \omega_0^2}{s^2(s^2 + a\omega_0^2)}
\]

Where \( a = 1 + \frac{m_L}{m_T} \)

d. From part c

\[
\frac{V_T}{F_T} = \frac{sX_T}{F_T} = \frac{1}{m_T} \frac{s^2 + \omega_0^2}{s(s^2 + a\omega_0^2)}
\]

Let \( F_T = \frac{F_0}{s} \) then

\[
V_T(s) = \frac{F_0}{m_T} \frac{s^2 + \omega_0^2}{s^2(s^2 + a\omega_0^2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + a\omega_0^2}
\]

After partial fraction expansions, so

\[
v_T(t) = A t + B + C \cos(a\omega_0 t + \theta)
\]

From which it is clear that \( v_T \xrightarrow{\text{as } t \to \infty} \infty \)

60.

a. Obtaining Laplace transforms on both sides of the equation

\[
sN(s) - N_0 = KN(s) \quad \text{or} \quad N(s) = \frac{N_0}{s - K}
\]

By inverse Laplace transformation

\[N(t) = N_0 e^{Kt}\]

b. Want to find the time at which

\[N_0 e^{Kt} = 2N_0\]

Obtaining \( \ln \) on both sides of the equation

\[t = \frac{\ln 2}{K}\]

61. The Laplace transform of the systems output is

\[
\mathcal{L}\{T(t)\} = T(s) = \frac{T_{\text{ref}}}{s} - \frac{T_{\text{ref}}}{s + \lambda} + \frac{2a\pi f}{s^2 + 4\pi^2 f^2} = \frac{T_{\text{ref}}}{s(s + \lambda)} + \frac{2a\pi f}{s^2 + 4\pi^2 f^2}
\]

Dividing by the input one gets
\[
\frac{T}{U}(s) = \frac{\lambda}{s + \lambda} + \frac{2a\pi f}{T_{ref}} \frac{s}{s^2 + 4\pi^2 f^2}
\]

62.

a. By direct differentiation
\[
\frac{dV(t)}{dt} = V_0 \frac{\lambda}{\alpha} (\alpha e^{-\alpha t}) e^{\frac{\lambda t}{\alpha}} = \lambda e^{-\alpha t} V(t)
\]

b. \( V(\infty) = \lim_{t \to \infty} V(t) = \lim_{t \to \infty} V_0 e^{\frac{\lambda t}{\alpha}} = V_0 e^{\frac{\lambda}{\alpha}} \)

c.

\[
\text{Lambda} = 2.5; \\
\alpha = 0.1; \\
V0 = 50; \\
t = linspace(0,100); \\
V = V0.*exp(Lambda.*(1-exp(-alpha.*t))/alpha);
\]

plot(t,V)
grid
xlabel('t (days)')
ylabel('mm^3 X 10^-3')
d. From the figure \( V(\infty) \approx 3.5 \times 10^{12} \text{ mm}^3 \times 10^{-3} \)

From part c \( V(\infty) = V_0 e^{k_1} = 50e^{0.1} = 3.6 \times 10^{12} \text{ mm}^3 \times 10^{-3} \)

63. Using the impedance method the two equations are:

\[ x_1 : \quad (ms^2 + k)x_1 - x_m k = F_1 \]
\[ x_m : \quad -x_1 k + (Bs + k)x_m = F_{iso} \]

Solving both equations simultaneously, one gets

\[
x_1 = \frac{\begin{vmatrix} F_1 & -k \\ F_{iso} & Bs + k \\ ms^2 + k & -k \\ -k & Bs + k \end{vmatrix}}{s} = \frac{F_1(Bs + k) + F_{iso}k}{s}\frac{F_1Bs + k(F_1 + F_{iso})}{(ms^2 + k)(Bs + k) - k^2} = \frac{F_1Bs + k(F_1 + F_{iso})}{s(ms^3 + kms + kB)}
\]

64. Opening the current source, we find the contribution of the voltage source, \( V_a(s) \), to the ac current, \( I_{acF_1}(s) \).

\[
I_{acF_1}(s) = \frac{V_a(s)}{Z(s)} = \frac{V_a(s)}{Ls + R + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1} V_a(s)
\]

Short-circuiting the voltage source, \( V_a(s) \), we find the contribution of the current source, \( I_{acR}(s) \), to the ac current, \( I_{acF_2}(s) \).

\[
I_{acF_2}(s) = I_{acR}(s) \frac{R + \frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{RCs + 1}{LCs^2 + RCs + 1} I_{acR}(s)
\]

Thus, the total current, \( I_{acF}(s) \), is given by:

\[
I_{acF}(s) = I_{acF_2}(s) + I_{acF_1}(s) = \frac{1 + RCs}{LCs^2 + RCs + 1} I_{acR}(s) + \frac{Cs}{LCs^2 + RCs + 1} V_a(s)
\]
65. Writing the loop equation around the armature circuit for the motor in Figure 2.35:

\[ e_a(t) = R_a i_a + L_a \frac{di_a}{dt} + K_p \frac{d\theta_m}{dt} \]

Taking the Laplace transform:

\[ E_a(s) = R_a I_a(s) + L_a sI_a(s) + K_p s\theta_m(s) \quad (1) \]

The torque developed at the motor is:

\[ T_m(t) = J_m \frac{d^2 \theta_m(t)}{dt^2} + D_m \frac{d\theta_m(t)}{dt} + K_m \theta_m(t) \]

Taking the Laplace transform:

\[ T_m(s) = J_m s^2 \theta_m(s) + D_m s \theta_m(s) + K_m \theta_m(s) = (J_m s^2 + D_m s + K_m) \theta_m(s) \]

But \( T_m(s) = K_I I_a(s) \). Solving for \( I_a(s) \) and substituting for \( T_m(s) \)

\[ I_a(s) = \frac{T_m(s)}{K_I} = \frac{1}{K_I} (J_m s^2 + D_m s + K_m) \theta_m(s) \]

Substituting in (1) for \( I_a(s) \) and simplifying

\[ E_a(s) = \frac{1}{K_I} \left[ (R_a + L_a s)(J_m s^2 + D_m s + K_m) + K_s K_b s \right] \theta_m(s) \]

Thus

\[ \frac{\theta_m(s)}{E_a(s)} = \frac{K_s}{J_m L_a s^3 + (J_m R_a + D_m L_a)s^2 + (D_m R_a + K_m L_a + K_s K_b)s + K_m R_a} \]

66.

a. Expressing \( \frac{A_c}{A} \sqrt{2gh} \) as a Taylor series around \( h_0 \)

\[ \frac{A_c}{A} \sqrt{2gh} \approx \frac{A_c}{A} \sqrt{2gh_0} + \frac{\partial}{\partial h} \left( \frac{A_c}{A} \sqrt{2gh} \right)_{h_0} \partial h = \frac{A_c}{A} \sqrt{2gh_0} + \frac{A_c}{A} \sqrt{2gh_0} \frac{g}{\partial h} \partial h \quad (1) \]

Also,

\[ h = h_0 + \partial h \quad (2) \]

and
\[ q = q_0 + \delta q \quad (3) \]

Substituting (1), (2), and (3) into the given nonlinear equation and eliminating the equilibrium values yields the linear equation

\[ \frac{d\delta h}{dt} + \frac{A_c}{A} \frac{g}{\sqrt{2gh_0}} \delta h = \frac{\delta q}{\rho A} \]

Thus the transfer function is

\[ H(s) = \frac{1/\rho A}{Q(s)} \left( s + \frac{A_c}{A} \frac{g}{\sqrt{2gh_0}} \right) \]

b. Substituting \( q_e = \rho A_c \sqrt{2gh_{aw}} \) into \( q - eq = \frac{d}{dt}(\rho A e h_{aw}) \)

\[ e_i q - e \rho A_c \sqrt{2gh_{aw}} = \rho A h_{aw} \frac{de}{dt} \]

Rearranging

\[ \rho A h_{aw} \frac{de}{dt} + e \rho A_c \sqrt{2gh_{aw}} = e_i q \]

Simplifying,

\[ \frac{de}{dt} + \frac{A_c}{Ah_{aw}} \sqrt{2gh_{aw}} e = e_i q \]

Taking the Laplace transform

\[ sE(s) + \frac{A_c}{Ah_{aw}} \sqrt{2gh_{aw}} E(s) = e_i Q(s) \]

From which,

\[ \frac{E(s)}{Q(s)} = \frac{e_i}{(s + \frac{A_c}{Ah_{aw}} \sqrt{2gh_{aw}})} \]
a. The first two equations are nonlinear because of the $Tv$ products on their right hand side. Otherwise the equations are linear.

b. To find the equilibria let \[
\frac{dT}{dt} = \frac{dT^*}{dt} = \frac{dv}{dt} = 0
\]

Leading to
\[
s - dT^* - \beta Tv = 0
\]
\[
\beta Tv - \mu T^* = 0
\]
\[
kT^* - cv = 0
\]

The first equilibrium is found by direct substitution. For the second equilibrium, solve the last two equations for $T^*$

\[
T^* = \frac{\beta Tv}{\mu} \text{ and } T^* = \frac{cv}{k}.
\]

Equating we get that $T = \frac{c\mu}{\beta k}$

Substituting the latter into the first equation after some algebraic manipulations we get that $v = \frac{ks}{c\mu}d - \frac{d}{\beta}$. It follows that $T^* = \frac{cv}{k} = \frac{s - cd}{\mu - k\beta}$.

68.

a. From $a = \frac{F - F_w}{k_m \cdot m}$, we have: $F = F_w + k_m \cdot m \cdot a = F_{R_0} + F_L + F_{St} + + k_m \cdot m \cdot a \ (1)$

Substituting for the motive force, $F$, and the resistances $F_{R_0}$, $F_L$, and $F_{St}$ using the equations given in the problem, yields the equation:

\[
F = \frac{P \cdot \eta_{ha}}{v} = f \cdot m \cdot g \cdot \cos \alpha + m \cdot g \cdot \sin \alpha + 0.5 \cdot \rho \cdot C_w \cdot A \cdot \left[ v + v_{hu} \right]^2 + k_m \cdot m \cdot a \ (2)
\]

b. Noting that constant acceleration is assumed, the average values for speed and acceleration are:

\[
a_{av} = 20 \text{ (km/h)}/ 4 \text{ s} = 5 \text{ km/h} \cdot \text{s} = 5 \times 1000/3600 \text{ m/s}^2 = 1.389 \text{ m/s}^2
\]
\[
v_{av} = 50 \text{ km/h} = 50,000/3,600 \text{ m/s} = 13.89 \text{ m/s}
\]

The motive force, $F$ (in N), and power, $P$ (in kW) can be found from eq. 2:

\[
F_{av} = 0.011 \times 1590 \times 9.8 + 0.5 \times 1.2 \times 0.3 \times 2 \times 13.89 + 1.2 \times 1590 \times 1.389 = 2891 \text{ N}
\]
\[ P_{av} = F_{av} \cdot \eta / \eta_{tot} = 2891 \times 13.89 / 0.9 = 44,617 \text{ N.m/s} = 44.62 \text{ kW} \]

To maintain a speed of 60 km/h while climbing a hill with a gradient \( \alpha = 5^\circ \), the car engine or motor needs to overcome the climbing resistance:

\[ F_{St} = m \cdot g \cdot \sin \alpha = 1590 \cdot 9.8 \cdot \sin 5^\circ = 1358 \text{ N} \]

Thus, the additional power, \( P_{add} \), the car needs after reaching 60 km/h to maintain its speed while climbing a hill with a gradient \( \alpha = 5^\circ \) is:

\[ P_{add} = F_{St} \cdot v / \eta = 1358 \times 60 \times 1000 / (3,600 \times 0.9) = 25,149 \text{ W} = 25.15 \text{ kW} \]

c. Substituting for the car parameters into equation 2 yields:

\[ F = 0.011 \times 1590 \times 9.8 + 0.5 \times 1.2 \times 0.3 \times 2 \cdot v^2 + 1.2 \times 1590 \ dv / dt \]

or \( F(t) = 171.4 + 0.36 \cdot v^2 + 1908 \ dv / dt \) (3)

To linearize this equation about \( v_o = 50 \text{ km/h} = 13.89 \text{ m/s} \), we use the truncated taylor series:

\[ v^2 - v_o^2 \approx \frac{d(v^2)}{dv} \bigg|_{v=v_o} (v - v_o) = 2v_o(v - v_o) \] (4), from which we obtain:

\[ v^2 = 2v_o \cdot v - v_o^2 = 27.78 \cdot v - 13.89^2 \] (5)

Substituting from equation (5) into (3) yields:

\[ F(t) = 171.4 + 10 \cdot v - 69.46 + 1908 \ dv / dt \] or

\[ F_e(t) = F(t) - F_{Ro} + F_o = F(t) - 171.4 + 69.46 = 10 \cdot v + 1908 \ dv / dt \] (6)

Equation (6) may be represented by the following block-diagram:
d. Taking the Laplace transform of the left and right-hand sides of equation (6) gives,

\[
F_e(s) = 10 V(s) + 1908 s V(s) \quad (7)
\]

Thus the transfer function, \( G_v(s) \), relating car speed, \( V(s) \) to the excess motive force, \( F_e(s) \), when the car travels on a level road at speeds around \( v_o = 50 \text{ km/h} = 13.89 \text{ m/s} \) under windless conditions is:

\[
G_v(s) = \frac{V(s)}{F_e(s)} = \frac{1}{10 + 1908 s} \quad (8)
\]

69.

a. Since the system’s transfer function exhibits a pure time delay of \( T \) seconds, the unit step response of the system is the unit step response of a first order system delayed \( T \) seconds, namely

\[
h(t) = K \left( 1 - e^{\frac{t-T}{\tau}} \right) u(t-T)
\]
c.

The output will be delayed $T$ seconds, thus writing

$$\frac{H(s)}{Q(s)} = \frac{K}{(1 + \tau s)} e^{-\tau s}$$

Then cross-multiplying

$$H(s)(1 + \tau s)e^{\tau s} = KQ(s)$$

And obtaining the inverse Laplace transform, one gets:

$$\tau \frac{d}{dt} h(t + T) + h(t + T) = Kq(t)$$