Solution Manual for Advanced Mechanics and General Relativity

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Chapter 1

Newtonian Gravity

Problem 1.1
From the Euler-Lagrange equations of motion:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0
\] (1.1)
we have
\[
m \ddot{x} + A = 0 \longrightarrow m \ddot{x} = -A,
\] (1.2)
and this corresponds to motion under the influence of a constant force \( A \) (for example, \( A = mg \)). The constant \( B \) provides an energy offset to the potential \( U = A x + B \), and has no dynamical effect.

Problem 1.2
a. For \( x(t) = \frac{x_f t}{T} \), we have:
\[
S = \int_0^T \frac{1}{2} m \dot{x}^2 \, dt = \int_0^T \frac{1}{2} m \left( \frac{x_f}{T} \right)^2 \, dt = \frac{1}{2} m \frac{x_f^2}{T}.
\] (1.3)

b. For \( x(t) = \frac{x_f t}{T} + \sum_{j=1}^{\infty} \alpha_j \sin \left( \frac{j \pi t}{T} \right) \) the action is:
\[
S = \frac{1}{2} m \frac{x_f^2}{T} + \frac{1}{2} m \int_0^T \left( \sum_{j=1}^{\infty} \alpha_j \frac{j \pi}{T} \cos \left( \frac{j \pi t}{T} \right) \right) \left( \sum_{k=1}^{\infty} \alpha_k \frac{k \pi}{T} \cos \left( \frac{k \pi t}{T} \right) \right) \, dt.
\] (1.4)

Noting that
\[
\int_0^T \cos \left( \frac{j \pi t}{T} \right) \cos \left( \frac{k \pi t}{T} \right) \, dt = \frac{1}{2} T \delta_{jk},
\] (1.5)
the sums in the second term of $S$ collapse:

$$S = \frac{1}{2} m \frac{x_j^2}{T} + \frac{1}{2} m \sum_{j=1}^{\infty} \frac{1}{2} T \left( \frac{j \pi}{T} \right)^2 \alpha_j^2$$

$$= \frac{1}{2} m \left( \frac{x_j^2}{T} + \frac{1}{2} T \sum_{j=1}^{\infty} \left( \frac{j \pi}{T} \right)^2 \alpha_j^2 \right).$$

(1.6)

The first term is the value of $S$ we got in part a. The second term, assuming it exists as an infinite sum, is a sum of positive quantities, so must itself be positive, and this action is larger than value in (1.3) for any non-zero $\alpha_j$.

**Problem 1.3**

The Lagrangian, written in cylindrical coordinates, is:

$$L = \frac{1}{2} m \left( s^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right) - U(s).$$

(1.7)

From the Euler-Lagrange equations, we have:

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{s}} \right) - \frac{\partial L}{\partial s} = \frac{d}{dt} (m \dot{s}) - m \ddot{s} - m s \ddot{\phi}^2 + \frac{dU}{ds} = m \ddot{s} - m s \ddot{\phi}^2 + \frac{dU}{ds}$$

(1.8)

so that

$$m \ddot{s} = m s \ddot{\phi}^2 - \frac{dU}{ds}.$$  

(1.9)

For $\phi$:

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = \frac{d}{dt} \left( m s^2 \dot{\phi} \right) = m s^2 \ddot{\phi} + 2 m s \dot{\phi}$$

(1.10)

or

$$m s^2 \ddot{\phi} = -2 m s \dot{\phi}.$$  

(1.11)

Finally, for $z$:

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = \frac{d}{dt} \left( m \dot{z} \right) = m \ddot{z}$$

(1.12)

so that

$$m \ddot{z} = 0.$$  

(1.13)
If we work from Newton’s second law in Cartesian coordinates, then we must use:

\[
x = s \cos \phi \quad \rightarrow \quad \ddot{x} = \cos \phi \dddot{s} - s \left( \cos \phi \dot{\phi}^2 + \sin \phi \dddot{\phi} \right) - 2 \sin \phi \dddot{s} \ddot{\phi} \\
y = s \sin \phi \quad \rightarrow \quad \ddot{y} = \sin \phi \dddot{s} + s \left( -\sin \phi \dot{\phi}^2 + \cos \phi \dddot{\phi} \right) + 2 \cos \phi \dddot{s} \ddot{\phi} \\
z = z \quad \rightarrow \quad \ddot{z} = \dddot{z}
\]

Together with Newton’s second law: \( ma = -\nabla U(\sqrt{x^2 + y^2}) \), in components:

\[
m \ddot{x} = -\frac{dU}{ds} \frac{x}{\sqrt{x^2 + y^2}} \\
m \ddot{y} = -\frac{dU}{ds} \frac{y}{\sqrt{x^2 + y^2}} \\
m \ddot{z} = 0
\]

Using the definitions, and derivatives in (1.14), we have:

\[
m \cos \phi \dddot{s} - m s \left( \cos \phi \dot{\phi}^2 + \sin \phi \dddot{\phi} \right) - 2 m \sin \phi \dddot{s} \ddot{\phi} = -\frac{dU}{ds} \cos \phi \\
m \sin \phi \dddot{s} + m s \left( -\sin \phi \dot{\phi}^2 + \cos \phi \dddot{\phi} \right) + 2 m \cos \phi \dddot{s} \ddot{\phi} = -\frac{dU}{ds} \sin \phi
\]

and if we multiply the top two equations by \( \sin \phi \) and \( \cos \phi \) respectively, we get:

\[
m \cos \phi \sin \phi \dddot{s} - m s \left( \cos \phi \sin \phi \dot{\phi}^2 + \sin^2 \phi \dddot{\phi} \right) - 2 m \sin^2 \phi \dddot{s} \ddot{\phi} = -\frac{dU}{ds} \cos \phi \sin \phi \\
m \sin \phi \cos \phi \dddot{s} + m s \left( -\sin \phi \cos \phi \dot{\phi}^2 + \cos^2 \phi \dddot{\phi} \right) + 2 m \cos^2 \phi \dddot{s} \ddot{\phi} = -\frac{dU}{ds} \sin \phi \cos \phi
\]

Subtracting the top from the bottom gives:

\[
m s \dddot{\phi} + 2 m \dddot{s} \ddot{\phi} = 0. \tag{1.18}
\]

Using this equation, solved for \( \dddot{\phi} \), in the top equation of (1.16) gives

\[
m \cos \phi \dddot{s} - m s \cos \phi \dot{\phi}^2 + 2 m \sin \phi \dddot{s} \dddot{\phi} - 2 m \sin \phi \dddot{s} \ddot{\phi} = -\frac{dU}{ds} \cos \phi \tag{1.19}
\]

from which we obtain

\[
m \dddot{s} - m s \dot{\phi}^2 = -\frac{dU}{ds}. \tag{1.20}
\]
Problem 1.4
Consider the Lagrangian for a central potential written in Cartesian coordinates:

\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(\sqrt{x^2 + y^2 + z^2}). \]  

(1.21)

To transform from Cartesian to cylindrical coordinates, we employ:

\[ x = s \cos(\phi), \quad y = s \sin(\phi), \quad \text{and} \quad z = z, \]  

(1.22)

and,

\[ \dot{x} = \dot{s} \cos(\phi) - s \sin(\phi) \dot{\phi}, \quad \text{and} \quad \dot{y} = \dot{s} \sin(\phi) + s \cos(\phi) \dot{\phi}. \]  

(1.23)

So, in particular,

\[ \dot{x}^2 = \cos^2(\phi) \dot{s}^2 - 2s \sin(\phi) \cos(\phi) \dot{s} \dot{\phi} + s^2 \sin^2(\phi) \dot{\phi}^2 \]  

and

\[ \dot{y}^2 = \sin^2(\phi) \dot{s}^2 + 2s \sin(\phi) \cos(\phi) \dot{s} \dot{\phi} + s^2 \cos^2(\phi) \dot{\phi}^2. \]  

(1.24)

Evidently, the kinetic term of the Lagrangian transforms via:

\[ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2, \]  

(1.25)

and the potential terms transforms via:

\[ x^2 + y^2 + z^2 = s^2 + z^2. \]  

(1.26)

So the Lagrangian for a central potential, written in cylindrical coordinates, is

\[ L = \frac{1}{2} m \left( \dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2 \right) - U\left( \sqrt{s^2 + z^2} \right). \]  

(1.27)

Setting \( x^1 = s, \ x^2 = \phi, \) and \( x^3 = z, \) the metric associated with cylindrical coordinates must be:

\[ g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]  

(1.28)

Problem 1.5
a. Consider the equation of motion:

\[ m g_{\alpha \nu} \ddot{x}^{\nu} + \frac{1}{2} m \dot{x}^{\nu} \dot{x}^{\gamma} \left( \frac{\partial g_{\alpha \nu}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha \gamma}}{\partial x^{\nu}} - \frac{\partial g_{\gamma \nu}}{\partial x^{\alpha}} \right) = - \frac{\partial U}{\partial x^{\alpha}}. \]  (1.29)

For \( \alpha = 1 \), corresponding to the \( r \) component, we have:

\[ m g_{1 \nu} \ddot{x}^{\nu} + \frac{1}{2} m \dot{x}^{\nu} \dot{x}^{\gamma} \left( \frac{\partial g_{1 \nu}}{\partial x^{\gamma}} + \frac{\partial g_{1 \gamma}}{\partial x^{\nu}} - \frac{\partial g_{\gamma \nu}}{\partial x^{1}} \right) = - \frac{\partial U}{\partial x^{1}}. \]  (1.30)

The metric, in spherical coordinates, is

\[ g_{\mu \nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}, \]  (1.31)

and the only non-zero terms are \( g_{11}, g_{22}, \) and \( g_{33} \). So, the \( r \) component of the equation of motion now reads:

\[ m \ddot{x}^{1} + \frac{1}{2} m \left( \ddot{x}^{2} \dot{x}^{2} \frac{\partial g_{22}}{\partial x^{1}} - \dot{x}^{2} \dot{x}^{3} \frac{\partial g_{33}}{\partial x^{1}} \right) = - \frac{\partial U}{\partial x^{1}}, \]  (1.32)

or, using \( x^1 = r, x^2 = \theta, \) and \( x^3 = \phi, \)

\[ m \ddot{r} + \frac{1}{2} m \left( -\dot{\theta}^2 2r - \dot{\phi}^2 2r \sin^2 \theta \right) = - \frac{\partial U}{\partial r}. \]  (1.33)

So,

\[ m \ddot{r} - \dot{r} \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 = - \frac{\partial U}{\partial r}. \]  (1.34)

b. Starting from

\[ L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - U(r), \]  (1.35)

we have:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0, \]  (1.36)

or with \( L \) inserted,

\[ \frac{d}{dt} (m \ddot{r}) - \frac{1}{2} m \left( 2r \dot{\theta}^2 + 2r \sin^2 \theta \dot{\phi}^2 \right) + \frac{\partial U}{\partial r} = 0. \]  (1.37)

So, just as in part a,

\[ m \ddot{r} - \dot{r} \dot{\theta}^2 + r \sin^2 \theta \dot{\phi}^2 = - \frac{\partial U}{\partial r}. \]  (1.38)
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Problem 1.6

The two-dimensional case suffices to show the pattern:

\[
\mathbf{A} \mathbf{x} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A^{11}x_1 + A^{12}x_2 \\ A^{21}x_1 + A^{22}x_2 \end{pmatrix} = \begin{pmatrix} A^{1j}x_j \\ A^{2j}x_j \end{pmatrix},
\]

so \( \mathbf{A} \mathbf{x} = A^{ij} x_j \). Similarly,

\[
x^T \mathbf{A} = (x_1 \ x_2) \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} x_1A^{11} + x_2A^{21} \\ x_1A^{12} + x_2A^{22} \end{pmatrix} = \begin{pmatrix} A^{j1}x_j \\ A^{j2}x_j \end{pmatrix},
\]

so \( x^T \mathbf{A} = A^{ji} x_j \).

Problem 1.7

a. Break up \( T_{\mu \nu} \) into pieces:

\[
T_{\mu \nu} = \frac{1}{2} (T_{\mu \nu} + T_{\nu \mu}) + \frac{1}{2} (T_{\mu \nu} - T_{\nu \mu}) \tag{1.41}
\]

and we can verify:

\[
S_{\nu \mu} = \frac{1}{2} (T_{\nu \mu} + T_{\mu \nu}) = S_{\mu \nu}
\]

\[
A_{\nu \mu} = \frac{1}{2} (T_{\nu \mu} - T_{\mu \nu}) = -A_{\mu \nu}. \tag{1.42}
\]

b. We have:

\[
T_{\mu \nu} Q^{\mu \nu} = S_{\mu \nu} Q^{\mu \nu} + A_{\mu \nu} Q^{\mu \nu}. \tag{1.43}
\]

But,

\[
A_{\mu \nu} Q^{\mu \nu} = -A_{\nu \mu} Q^{\mu \nu} = -A_{\nu \mu} Q^{\mu \nu} = -A_{\mu \nu} Q^{\mu \nu} \implies A_{\mu \nu} Q^{\mu \nu} = 0. \tag{1.44}
\]

So,

\[
T_{\mu \nu} Q^{\mu \nu} = S_{\mu \nu} Q^{\mu \nu}. \tag{1.45}
\]

Similarly,

\[
T_{\mu \nu} P^{\mu \nu} = S_{\mu \nu} P^{\mu \nu} + A_{\mu \nu} P^{\mu \nu} = A_{\mu \nu} P^{\mu \nu}. \tag{1.46}
\]
c. Let \( Q^{\nu\gamma} \equiv \dot{x}^\nu \dot{x}^\gamma \), a symmetric tensor, and

\[
T_{\alpha\nu\gamma} = \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\alpha\nu}}{\partial x^\alpha}.
\]

(1.47)

From part b, \( Q^{\nu\gamma} T_{\alpha\nu\gamma} = Q^{\nu\gamma} S_{\alpha\nu\gamma} \), where \( S_{\alpha\nu\gamma} \) is the portion of \( T_{\alpha\nu\gamma} \) symmetric under \( \nu \leftrightarrow \gamma \) interchange. From part a,

\[
S_{\alpha\nu\gamma} = \frac{1}{2} (T_{\alpha\nu\gamma} + T_{\alpha\gamma\nu})
\]

\[
= \frac{1}{2} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\alpha\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{1}{2} \frac{\partial g_{\alpha\gamma}}{\partial x^\alpha} \right),
\]

and since \( g_{\gamma\nu} = g_{\nu\gamma} \),

\[
= \frac{1}{2} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\nu\gamma}}{\partial x^\alpha} \right).
\]

(1.48)

Hence,

\[
\dot{x}^\nu \dot{x}^\gamma \left( \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\nu\gamma}}{\partial x^\alpha} \right) = \frac{1}{2} \dot{x}^\nu \dot{x}^\gamma \left( \frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\nu\gamma}}{\partial x^\alpha} \right).
\]

(1.49)

Problem 1.8

a. We cannot use \( \rho(\phi) \) or \( r(\phi) \) for purely radial motion since in the case of, for example, infall along the \( \hat{y} \) axis, \( \phi = \pi/2 \), a constant, leaving no varying parameter to describe the evolution of \( r \).

b. We must go back to the temporally parameterized Lagrangian:

\[
L = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - U(r).
\]

(1.50)

Rather than developing the equations of motion directly from \( L \), we will work this time from the Hamiltonian \( H \) (which is numerically equivalent to the total energy \( E \)) and its conservation; we have

\[
H = E = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) + U(r).
\]

(1.51)

If we set \( \phi = 0 \), so the radial infall occurs along the \( \hat{x} \) axis (say), then

\[
E = \frac{1}{2} m r^2 - \frac{G M m}{r}.
\]

(1.52)
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We start from spatial infinity at rest, \( r(-\infty) = \infty \) and \( \dot{r}(-\infty) = 0 \), so the total energy for this trajectory is (and will remain) \( E = 0 \). We are left to solve:

\[
\dot{r}^2 = \frac{2GM}{r} \implies \dot{r} = -\sqrt{\frac{2GM}{r}}. \tag{1.53}
\]

To solve, we separate variables and integrate with respect to \( t \). On the left:

\[
\int_0^t \sqrt{\dot{r} \dot{t}} \, dt = \int_R^{r(t)} \sqrt{r} \, dr = \frac{2}{3} \left( r(t)^{3/2} - R^{3/2} \right), \tag{1.54}
\]

and on the right,

\[
- \int_0^t \sqrt{2MG} \, dt = -\sqrt{2MG} t. \tag{1.55}
\]

Equating and solving for \( r(t) \) gives our solution for radial infall:

\[
r(t) = \left( R^{3/2} - 3\sqrt{\frac{MG}{2}} t \right)^{2/3}. \tag{1.56}
\]

Problem 1.9

a. For

\[
L = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) + \frac{M}{r}, \tag{1.57}
\]

we have the Hamiltonian:

\[
H = \frac{1}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - \frac{M}{r}. \tag{1.58}
\]

Using the \( \phi \) component of the equation of motion:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \implies r^2 \dot{\phi} = J_z, \tag{1.59}
\]

we can rewrite \( H \) as:

\[
H = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \frac{J_z^2}{r^4} - \frac{M}{r}. \tag{1.60}
\]

Then identify the last two terms as the effective potential:

\[
U_{\text{eff}}(r) = \frac{1}{2} \frac{J_z^2}{r^2} - \frac{M}{r}. \tag{1.61}
\]
A sketch is shown in Figure 1.1. This has a zero at

$$U_{\text{eff}} = 0 \implies \frac{1}{2} \frac{J_z^2}{r_0^2} = \frac{M}{r_0} \implies r_0 = \frac{J_z^2}{2M}. \quad (1.62)$$

and a minimum at

$$\frac{\partial U_{\text{eff}}}{\partial r} = 0 \implies -\frac{J_z^2}{r_{\text{min}}^3} + \frac{M}{r_{\text{min}}^2} = 0 \implies r_{\text{min}} = \frac{J_z^2}{M}. \quad (1.63)$$

b. First,

$$U_{\text{min}} = U_{\text{eff}}(J_z^2/M) = \frac{1}{2} \frac{M^2}{J_z^2} - \frac{M^2}{J_z^2} = -\frac{1}{2} \frac{M^2}{J_z^2}. \quad (1.64)$$

Then, returning to $H = E = U_{\text{min}},$

$$-\frac{1}{2} \frac{M^2}{J_z^2} = \frac{1}{2} r^2 + \frac{1}{2} \frac{r^2 J_z^2}{r^4} - \frac{M}{r} \implies r^2 = 0. \quad (1.65)$$

We are left with a quadratic in $r$:

$$\frac{1}{2} J_z^2 - Mr + \frac{1}{2} \frac{M^2}{J_z^2} r^2 = 0, \quad (1.66)$$

and the solutions are:

$$r = \frac{M \pm \sqrt{M^2 - 4(J_z^2/2)(M^2/2J_z^2)}}{2(M^2/2J_z^2)} = \frac{J_z^2}{M} = r_{\text{min}}. \quad (1.67)$$
So the orbit is a circle with radius \( r = J_z^2 / M \).

**Problem 1.10**

a. For

\[
    r(\phi) = \frac{p}{1 + e \cos \phi},
\]

we have

\[
\begin{array}{c|c}
    (p, e) = (1, 1/2): & (p, e) = (1/2, 1/2): & (p, e) = (1, 1/4): \\
    \phi & r(\phi) & \phi & r(\phi) & \phi & r(\phi) \\
    0 & \frac{1}{1+1/2} = 2/3 & 0 & \frac{1/2}{1+1/2} = 1/3 & 0 & \frac{1}{1+1/4} = 4/5 \\
    \pi/2 & 1 & \pi/2 & 1/2 & \pi/2 & 1 \\
    \pi & \frac{1}{1-1/2} = 2 & \pi & \frac{1/2}{1-1/2} = 1 & \pi & \frac{1}{1-1/4} = 4/3
\end{array}
\]

Sketches are shown in Figure 1.2. A circle of radius \( R \) has \((p, e) = (R, 0)\).

![Figure 1.2: Sketch of ellipses defined by \( p \) and \( e \).](image)

b. The closest and furthest approach correspond to \( \phi = 0 \) and \( \phi = \pi \), respectively. So,

\[
    r_p = \frac{p}{1 + e} \quad \text{and} \quad r_a = \frac{p}{1 - e},
\]

(1.69)
and then,

\[ p = (1 + e) r_p \implies r_a = \frac{1 + e}{1 - e} r_p \implies r_a - e(r_a + r_p) = r_p. \quad (1.70) \]

Hence,

\[ e = \frac{r_a - r_p}{r_a + r_p} \quad \text{and} \quad p = (1 + e) r_p = \frac{2r_a r_p}{r_a + r_p}. \quad (1.71) \]

c. We have:

\[ r(\phi) = \frac{1}{M/J_z^2 + \alpha \cos \phi} = \frac{J_z^2 / M}{1 + (\alpha J_z^2 / M) \cos \phi}. \quad (1.72) \]

So here,

\[ e = \frac{\alpha J_z^2}{M} \quad \text{and} \quad p = \frac{J_z^2}{M}. \quad (1.73) \]

Then, using the result from part b,

\[ \frac{r_a - r_p}{r_a + r_p} = \frac{\alpha J_z^2}{M} = \alpha \left( \frac{2r_a r_p}{r_a + r_p} \right) \implies \alpha = \frac{r_a - r_p}{2r_a r_p}, \quad (1.74) \]

and

\[ \frac{2r_a r_p}{r_a + r_p} = \frac{J_z^2}{M} \implies J_z = \pm \sqrt{\frac{2Mr_a r_p}{r_a + r_b}}. \quad (1.75) \]

**Problem 1.11**

a. For the transformation: \( x \to \bar{x} = \bar{x}(x) \), the coordinate differential for \( \bar{x} \) is, by the chain rule:

\[ dx^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} dx^\beta. \quad (1.76) \]

So, \( dx \) transforms as a contravariant 1st rank tensor.

A scalar \( \phi(x) \) responds to \( x \to \bar{x} \) by transcription (i.e. not at all):

\[ \bar{\phi}(\bar{x}) = \phi(x(\bar{x})). \quad (1.77) \]

The derivatives of \( \bar{\phi}(\bar{x}) \) with respect to \( \bar{x} \) (forming the gradient) can be written in terms of the derivatives of \( \phi(x) \) with respect to \( x \):

\[ \frac{\partial \bar{\phi}}{\partial \bar{x}^\mu} = \frac{\partial \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^\mu}, \quad (1.78) \]
or using $\phi_{,\mu} \equiv \frac{\partial \phi}{\partial x^\mu}$,

$$\phi_{,\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \phi_{,\alpha}.$$  \hfill (1.79)

So, $\phi_{,\mu}$ transforms as a covariant 1st rank tensor.

b. For:

$$x = s \cos \phi, \quad dx = ds \cos \phi - s \sin \phi \, d\phi$$  \hfill (1.80)
$$y = s \sin \phi, \quad dy = ds \sin \phi + s \cos \phi \, d\phi,$$  \hfill (1.81)

the matrix relation between differentials is:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos \phi & -s \sin \phi \\ \sin \phi & s \cos \phi \end{pmatrix} \begin{pmatrix} ds \\ d\phi \end{pmatrix},$$

$$\approx \Lambda.$$  \hfill (1.82)

or going the other direction,

$$\begin{pmatrix} ds \\ d\phi \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\frac{1}{s} \sin \phi & \frac{1}{s} \cos \phi \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix},$$

$$\approx \Lambda^{-1}.$$  \hfill (1.83)

From the coordinate differential transformation (contravariant 1st rank tensor) we expect:

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} dx^\beta.$$  \hfill (1.84)

But is $\frac{\partial x^\alpha}{\partial \bar{x}^\beta}$ really the object representing $\Lambda^{-1}$? To form $\frac{\partial x^\alpha}{\partial \bar{x}^\beta}$, we need $\bar{x}^\alpha(x)$:

$$s = \sqrt{x^2 + y^2}, \quad \text{and} \quad \phi = \tan^{-1}(y/x).$$  \hfill (1.85)

Then,

$$\frac{\partial s}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial s}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2}.$$  \hfill (1.86)

To make the comparison with $\Lambda^{-1}$, we need to express $\frac{\partial x^\alpha}{\partial \bar{x}^\beta}$, i.e. the derivatives above, in terms of $s$ and $\phi$ rather than $x$ and $y$. Doing so, we find that

$$\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \left( \begin{array}{c} \frac{\partial s}{\partial x} \\ \frac{\partial s}{\partial \phi} \\ \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{array} \right) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\frac{1}{s} \sin \phi & \frac{1}{s} \cos \phi \end{pmatrix} = \Lambda^{-1}.$$  \hfill (1.87)
c. For \( \psi = kxy \), we have:

\[
\psi_{,\mu} = \begin{pmatrix}
\frac{\partial \psi}{\partial x}
\frac{\partial \psi}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
k y
k x
\end{pmatrix}.
\] (1.88)

Or, using \( x = s \cos \phi \) and \( y = s \sin \phi \), \( \tilde{\psi} = ks^2 \cos \phi \sin \phi \), and we have:

\[
\tilde{\psi}_{,\mu} = \begin{pmatrix}
\frac{\partial \tilde{\psi}}{\partial x}
\frac{\partial \tilde{\psi}}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
2ks \cos \phi \sin \phi 
k s^2 (-\sin^2 \phi + \cos^2 \phi)
\end{pmatrix}.
\] (1.89)

To check that \( \tilde{\psi}_{,\mu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \psi_{,\alpha} \), we must express \( \tilde{\psi}_{,\mu} \) in terms of \( s \) and \( \phi \):

\[
\tilde{\psi}_{,\mu}(s, \phi) = \begin{pmatrix}
k s \sin \phi
k s \cos \phi
\end{pmatrix}.
\] (1.90)

Then, the right side of the covariant transformation rule reads:

\[
\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \psi_{,\alpha} = \begin{pmatrix}
\frac{\partial x}{\partial x^1} & \frac{\partial x}{\partial x^2}
\frac{\partial y}{\partial x^1} & \frac{\partial y}{\partial x^2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \psi}{\partial x^1}
\frac{\partial \psi}{\partial x^2}
\end{pmatrix}
= \begin{pmatrix}
\cos \psi (ks \sin \phi) + \sin \phi (ks \cos \phi)
- \sin \phi (ks \sin \phi) + \cos \phi (ks \cos \phi)
\end{pmatrix}
= \begin{pmatrix}
2ks \cos \phi \sin \phi 
k s^2 (-\sin^2 \phi + \cos^2 \phi)
\end{pmatrix},
\] (1.91)

precisely \( \tilde{\psi}_{,\mu} \) from above.

**Problem 1.12**

For \( x^1 = x \), \( x^2 = y \), \( \bar{x}^1 = s \), and \( \bar{x}^2 = \phi \), with \( x = s \cos \phi \), \( y = s \sin \phi \), or \( s = \sqrt{x^2 + y^2} \), \( \phi = \tan^{-1}(y/x) \), we have:

\[
\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \equiv \begin{pmatrix}
\frac{\partial x}{\partial \bar{x}^1} & \frac{\partial x}{\partial \bar{x}^2}
\frac{\partial y}{\partial \bar{x}^1} & \frac{\partial y}{\partial \bar{x}^2}
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & - s \sin \phi
-s \sin \phi & s \cos \phi
\end{pmatrix}
= \begin{pmatrix}
\frac{x}{\sqrt{x^2 + y^2}} & -y
-y & \frac{y}{\sqrt{x^2 + y^2}}
\end{pmatrix}
\] (1.92)

in terms of \( \bar{x}^\alpha \)

and,

\[
\frac{\partial x^\alpha}{\partial \bar{x}^\mu} = \begin{pmatrix}
\frac{\partial x}{\partial \bar{x}^1} & \frac{\partial x}{\partial \bar{x}^2}
\frac{\partial y}{\partial \bar{x}^1} & \frac{\partial y}{\partial \bar{x}^2}
\end{pmatrix}
= \begin{pmatrix}
\sqrt{x^2 + y^2} & \frac{x}{\sqrt{x^2 + y^2}}
\frac{y}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}}
\end{pmatrix}
= \begin{pmatrix}
\cos \phi & \sin \phi
-s \sin \phi & s \cos \phi
\end{pmatrix}
\] (1.93)

in terms of \( x^\alpha \).
Now, in \( x^\alpha \) coordinates:

\[
\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\gamma} = \begin{pmatrix}
\frac{x}{\sqrt{x^2+y^2}} & -y & x \\
\frac{y}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & \frac{y}{x} \\
x & \frac{x}{\sqrt{x^2+y^2}} & x
\end{pmatrix}
\]

\[
\frac{\partial \bar{x}^\beta}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^\gamma} = \begin{pmatrix}
\cos \phi & -s \sin \phi & \sin \phi \\
\sin \phi & s \cos \phi & s \\
-s & 0 & 0
\end{pmatrix}
\]

and in \( \bar{x}^\alpha \) coordinates:

\[
\frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\gamma} = \begin{pmatrix}
\cos \phi & -s \sin \phi & \sin \phi \\
\sin \phi & s \cos \phi & s \\
-s & 0 & 0
\end{pmatrix}
\]

\[
\frac{\partial \bar{x}^\beta}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^\gamma} = \begin{pmatrix}
\cos \phi & -s \sin \phi & \sin \phi \\
\sin \phi & s \cos \phi & s \\
-s & 0 & 0
\end{pmatrix}
\]

Problem 1.13

a. The 2\textsuperscript{nd} rank contravariant tensor made from the direct product of two first rank contravariant tensors, \( T^{\mu\nu} = f^\mu h^\nu \), transforms as:

\[
T^{\mu\nu} = \tilde{f}^\mu \tilde{h}^\nu = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} f^\alpha \frac{\partial \tilde{x}^\nu}{\partial x^\beta} h^\beta
\]

\[
= \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} f^\alpha \frac{\partial \tilde{x}^\nu}{\partial x^\beta} h^\beta
\]

\[
\tilde{T}^{\mu\nu} = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} T^{\alpha\beta}.
\]

b. The 2\textsuperscript{nd} rank covariant tensor made from the direct product of two first rank covariant tensors, \( T_{\mu\nu} = f_\mu h_\nu \), transforms as:

\[
\bar{T}_{\mu\nu} = \bar{f}_\mu \bar{h}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} f^\alpha h^\beta
\]

\[
= \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} f^\alpha h^\beta
\]

\[
\bar{T}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} T_{\alpha\beta}.
\]

c. For contravariant and covariant tensors, \( f^\alpha \) and \( h_\beta \), we form \( \psi = f^\alpha h_\alpha \). \( \psi \) transforms as:

\[
\tilde{\psi} = \tilde{f}^\alpha \tilde{h}_\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} f^\mu \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} h_\gamma
\]

\[
= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\alpha} f^\mu h_\gamma
\]

\[
= \delta^\gamma_\mu f^\mu h_\gamma = \delta^\gamma_\gamma f^\gamma h_\gamma = \psi.
\]

So \( \psi \) transforms as \( \tilde{\psi} = \psi \), i.e. a scalar.
Problem 1.14
To show that a matrix is uninvertible, we need to show that the matrix determinant is zero. For $A_{\mu \nu}$ constructed from $p_{\mu}, q_{\nu}$, the matrix form looks like:

$$A_{\mu \nu} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$$

and the determinant of this matrix is:

$$\det A = (p_1 q_1) (p_2 q_2) - (p_1 q_2) (p_2 q_1) = 0$$

so that $A$ is uninvertible.

Problem 1.15
a. For $f(x) = \sin(x)$, we have:

$$p = \frac{df}{dx} = \cos(x) \rightarrow x = \arccos(p).$$

From the definition of the Legendre transformation, we have:

$$g(p) = px(p) - f(p)$$
and \( f(p) = f(x(p)) = \sqrt{1 - p^2} \), so that

\[
g(p) = p \arccos(p) - \sqrt{1 - p^2}.
\]

(1.106)

b. Starting with \( g(p) = p \arccos(p) - \sqrt{1 - p^2} \), we have:

\[
x = \frac{dg}{dp} = \arccos(p) \rightarrow p = \cos(x)
\]

(1.107)

and the Legendre transform of \( g(p) \) is:

\[
h(x) = p(x) x - g(x)
\]

(1.108)

where \( g(x) = x \cos(x) - \sqrt{1 - \cos^2 x} \), so

\[
h(x) = \cos(x) x - \left( x \cos(x) - \sqrt{1 - \cos^2 x} \right) = \sin(x)
\]

(1.109)

our starting point, \( f(x) \).

**Problem 1.16**

Using the transformation associated with \( K(x, \vec{x}) \):

\[
p = \frac{\partial K}{\partial x} \quad \vec{p} = -\frac{\partial K}{\partial \vec{x}}
\]

(1.110)

we can set \( p = \vec{x} \) and \( \vec{p} = -x \) by taking \( K(x, \vec{x}) = x \vec{x} \). This is a canonical transformation, so \( H(\vec{x}, \vec{p}) = H(x, \vec{x}, p, \vec{p}) \):

\[
\vec{H} = \frac{\vec{x}^2}{2m} + \frac{1}{2} k \vec{p}^2.
\]

(1.111)

**Problem 1.17**

For \( L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + \frac{G M m}{x} \)

a. The equation of motion is:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = m \ddot{x} + \frac{G M m}{x^2} = 0
\]

(1.112)

giving

\[
m \ddot{x} = -\frac{G M m}{x^2}.
\]

(1.113)
b. The definition of $p$ here is:

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \tag{1.114}$$

and the Legendre transform of this $L$ is:

$$H = p \dot{x}(p) - L(x, p) = \frac{p^2}{m} - \frac{1}{2} \frac{p^2}{m} - \frac{G M m}{x} = \frac{1}{2} \frac{p^2}{m} - \frac{G M m}{x}. \tag{1.115}$$

The two associated equations of motion are:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{1.116}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\left(\frac{G M m}{x^2}\right).$$

Solving the top equation for $p$ and inserting in the bottom we recover:

$$m \ddot{x} = -\frac{G M m}{x^2}. \tag{1.117}$$

**Problem 1.18**

The Lagrangian is:

$$L = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} - (q \phi - q \mathbf{v} \cdot \mathbf{A}). \tag{1.118}$$

The canonical momentum $p = \frac{\partial L}{\partial \mathbf{v}}$, so:

$$p = m \mathbf{v} + q \mathbf{A} \rightarrow \mathbf{v} = \frac{p - q \mathbf{A}}{m}. \tag{1.119}$$

Inserting this in the Legendre transform of $L$ serves to define $H$:

$$H = p \cdot \mathbf{v} - L$$

$$= \frac{1}{m} p \cdot (p - q \mathbf{A}) - \frac{1}{2 m} (p - q \mathbf{A}) \cdot (p - q \mathbf{A}) + q \phi - \frac{q}{m} \mathbf{A} \cdot (p - q \mathbf{A})$$

$$= \frac{1}{m} (p - q \mathbf{A}) \cdot (p - q \mathbf{A}) - \frac{1}{2 m} (p - q \mathbf{A}) \cdot (p - q \mathbf{A}) + q \phi$$

$$H = \frac{1}{2 m} (p - q \mathbf{A}) \cdot (p - q \mathbf{A}) + q \phi. \tag{1.120}$$
Problem 1.19
We have $[H, H] = 0$. What are $f^\alpha$ and $h_\alpha$ in the infinitesimal transformation:

$$\bar{x}^\alpha = x^\alpha + \epsilon f^\alpha \quad \bar{p}_\alpha = p_\alpha + \epsilon h_\alpha,$$

when $H$ is the generator?

We have

$$f^\alpha(x, p) = \frac{\partial H}{\partial p_\alpha}, \quad h_\alpha(x, p) = -\frac{\partial H}{\partial x^\alpha}. \quad (1.122)$$

We know from the equations of motion that

$$\frac{\partial H}{\partial p_\alpha} = \dot{x}^\alpha(t) \quad \text{and} \quad \frac{\partial H}{\partial x^\alpha} = -\dot{p}_\alpha(t), \quad (1.123)$$

so

$$\bar{x}^\alpha(t) = x^\alpha(t) + \epsilon \dot{x}^\alpha(t) \quad \text{and} \quad \bar{p}_\alpha(t) = p_\alpha(t) + \epsilon \dot{p}_\alpha(t). \quad (1.124)$$

Then the new coordinates are, to $O(\epsilon^2)$, (and hence, beyond our interest):

$$\bar{x}^\alpha(t) = x^\alpha(t + \epsilon) \quad \bar{p}_\alpha(t) = p_\alpha(t + \epsilon), \quad (1.125)$$

i.e. the transformation generated by $H$ produces $\bar{x}(t)$ and $\bar{p}(t)$ that are $x(t)$ and $p(t)$ propagated forward in time (for $\epsilon > 0$).

Problem 1.20
The only components not specifically worked out following (1.180) involve $f_{1;\mu}$ and $f_{\mu;1}$—since $f_1 = 0$, we know that $f_{1;1} = 0$. The second remaining element is:

$$f_{1;2} + f_{2;1} = \underbrace{f_{1;2}}_{\Gamma^\sigma_{12} f_\sigma + f_{2;1} - \Gamma^\sigma_{21} f_\sigma} = 0. \quad (1.126)$$

(dropping the factor of 1/2 that comes from symmetrization, for convenience) and $\Gamma^\theta_{r\theta} = \frac{1}{r}$ (the rest are zero), so we have:

$$f_{1;2} + f_{2;1} = -\frac{1}{r} \left(r^2 f^\theta\right) + \frac{\partial}{\partial r} \left(r^2 f^\theta\right) - \frac{1}{r} \left(r^2 f^\theta\right) = 0 \quad (1.127)$$

and this becomes, after cancellations:

$$r^2 \frac{\partial}{\partial r} f^\theta = 0 \quad (1.128)$$
which tells us that \( f^\theta \) will be independent of \( r \) (as it is).
The final component is:

\[
 f_{1;3} + f_{3;1} = f_{1,3} - \Gamma_{13}^\sigma f_\sigma + f_{3,1} - \Gamma_{13}^\sigma f_\sigma, \tag{1.129}
\]

and the only non-zero connection coefficient here is \( \Gamma^\phi_{r\phi} = \frac{1}{r} \), so:

\[
 f_{1;3} + f_{3;1} = - \frac{2}{r} \left( r^2 \sin^2 \theta \ f^\phi \right) + \frac{\partial}{\partial r} \left( r^2 \sin^2 \theta \ f^\phi \right) = 0, \tag{1.130}
\]

and again, using the product rule, this gives:

\[
 r^2 \sin^2 \theta \frac{\partial}{\partial r} f^\phi = 0 \tag{1.131}
\]

indicating that \( f^\phi \) is also \( r \)-independent.

**Problem 1.21**

a. In Cartesian coordinates,

\[
 f_{\mu;\nu} = f_{\mu,\nu} - \Gamma^\sigma_{\mu\nu} f_\sigma = f_{\mu,\nu} \tag{1.132}
\]

since the metric has no spatial dependence:

\[
 \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) = 0. \tag{1.133}
\]

Killing’s equation then reads:

\[
 f_{\mu;\nu} + f_{\nu;\mu} = 0 = f_{\mu,\nu} + f_{\nu,\mu}. \tag{1.134}
\]

For \( \bar{x} = x + \omega \hat{\Omega} \times x \), or in index notation,

\[
 \bar{x}^\alpha = x^\alpha + \omega g^{\alpha\beta} \epsilon_\beta_{\mu\nu} \Omega^\mu x^\nu,
\]

we have

\[
 f^\alpha = g^{\alpha\beta} \epsilon_\beta_{\mu\nu} \Omega^\mu x^\nu \quad \text{and} \quad f_\alpha = \epsilon_{\alpha\mu\nu} \Omega^\mu x^\nu. \tag{1.135}
\]

Then,

\[
 f_{\alpha,\beta} = \frac{\partial f_\alpha}{\partial x^\beta} = \epsilon_{\alpha\mu\nu} \Omega^\mu \delta^\nu_\beta = \epsilon_{\alpha\beta\mu} \Omega^\mu, \tag{1.136}
\]

and finally, using the fact that \( \epsilon_{\alpha\beta\mu} = -\epsilon_{\beta\alpha\mu} \),

\[
 f_{\mu,\nu} = \epsilon_{\alpha\beta\mu} \Omega^\mu + \epsilon_{\beta\alpha\mu} \Omega^\mu = \epsilon_{\alpha\beta\mu} \Omega^\mu - \epsilon_{\alpha\beta\mu} \Omega^\mu = 0, \tag{1.137}
\]
CHAPTER 1. NEWTONIAN GRAVITY

So Killing’s equation is true and and \( f^\alpha \) is a Killing vector.

b. For \( f^\alpha \) as given in part a, we have \( J = p_\alpha f^\alpha = p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^\mu x^\nu \) and

\[
\bar{p}_\gamma = p_\gamma - \omega \frac{\partial J}{\partial x^\gamma} = p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^\mu \delta^\nu_\gamma
\]

\[
= p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^\mu.
\]

(1.139)

Then,

\[
\bar{p}_\alpha g^{\gamma\nu} \bar{p}_\nu = \left( p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^\mu \right) \left( p^\gamma - \omega p_\alpha g^{\alpha\beta} g^{\gamma\rho} \epsilon_{\beta\mu\rho} \Omega^\mu \right)
\]

\[
= p_\gamma p^\gamma - \omega \left[ p_\gamma p_\alpha g^{\alpha\beta} g^{\gamma\rho} \epsilon_{\beta\mu\rho} \Omega^\mu + p^\gamma p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^\mu \right] + O(\omega^2)
\]

\[
= p_\gamma p^\gamma - \omega \left[ p^\rho p^\beta \epsilon_{\beta\mu\rho} \Omega^\mu + p^\gamma p^\beta \epsilon_{\beta\mu\gamma} \Omega^\mu \right] + O(\omega^2)
\]

the product of symmetric and antisymmetric tensors is zero, so each term in the bracket dies, and

\[
= p_\gamma g^{\gamma\alpha} p_\alpha + O(\omega^2).
\]

(1.140)

For \( r^2 \),

\[
\tilde{r}^2 = \bar{x}^\alpha g_{\alpha\beta} \bar{x}^\beta = \left( x^\alpha + \omega g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^\mu x^\nu \right) \left( x_\alpha + \omega \epsilon_{\beta\mu\nu} \Omega^\mu x^\nu \right)
\]

\[
= x^\alpha x_\alpha + \omega \left[ x^\alpha x^\nu \epsilon_{\alpha\mu\nu} \Omega^\mu + g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^\mu x^\nu x_\alpha \right] + O(\omega^2)
\]

\[
= x^\alpha x_\alpha + O(\omega^2).
\]

(1.141)

It follows that,

\[
U(\tilde{r}) = U(r) + O(\omega^2).
\]

(1.142)

Thus,

\[
\tilde{H} = \frac{1}{2m} \bar{p}_\alpha \bar{p}^\alpha + U(\tilde{r}) = \frac{1}{2m} p_\alpha p^\alpha + U(r) + O(\omega^2) = H + O(\omega^2).
\]

(1.143)

**Problem 1.22**

If the target is

\[
\bar{x}^\alpha = x^\alpha + \epsilon F^\alpha(p)
\]

(1.144)

then we must have:

\[
\frac{\partial J(x, p)}{\partial p_\alpha} = F^\alpha.
\]

(1.145)
Whatever $J$ is, it cannot depend on $x$, by assumption, since $\bar{p}_\beta = p_\beta - \epsilon \frac{\partial J}{\partial x^\alpha}$, and we are given $\bar{p}_\beta = p_\beta$. The Poisson bracket of $H$ with $J$ reads:

$$[H, J] = \frac{\partial H}{\partial x^\alpha} \frac{\partial J}{\partial p_\alpha} - \frac{\partial H}{\partial p_\alpha} \frac{\partial J}{\partial x^\alpha}$$

(1.146)

and the second term is zero, so we must have:

$$\frac{\partial H}{\partial x^\alpha} F^\alpha(p) = 0.$$  

(1.147)

Without an explicit relationship between $x^\alpha$ and $p_\alpha$, there is no way to make this zero beyond the geometrical requirement: $\nabla H \cdot F = 0$. So, orthogonality is required in this case. In one dimension, this is unavailable, and we must have $F = 0$, then $J = \text{constant}$ and there is no transformation.

**Problem 1.23**

Starting with

$$L = \frac{1}{2} m v_j v^j - (q \phi - q v_j A^j),$$

(1.148)

we have the Euler-Lagrange equations:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v_j} \right) - \frac{\partial L}{\partial x^j} = 0.$$  

(1.149)

Since we are in Cartesian coordinates, there is no distinction between up and down indices, and the above reads:

$$m \ddot{v}_j + q \frac{\partial A^j}{\partial x^k} v_k + q \frac{\partial \phi}{\partial x^j} - q v_k \frac{\partial A^k}{\partial x^j} = 0.$$  

(1.150)

We can write this as

$$m \ddot{x}^j = -q \frac{\partial \phi}{\partial x^j} + q v_k \left( \frac{\partial A^k}{\partial x^j} - \frac{\partial A^j}{\partial x^k} \right).$$  

(1.151)

Note, either by working out components, or using the Levi-Civita form for $v \times (\nabla \times A)$:

$$\epsilon_{ijk} v_j \left( \epsilon_{k\ell m} \frac{\partial}{\partial x^\ell} A^m \right) = \epsilon_{ijk} \epsilon_{k\ell m} v_j \frac{\partial}{\partial x^\ell} A^m$$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell} ) v_j \frac{\partial}{\partial x^\ell} A^m$$

$$= v_j \frac{\partial A_j}{\partial x^i} - v_j \frac{\partial A^i}{\partial x^j}.$$  

(1.152)
that the second term in (1.151) can be written $q \mathbf{v} \times (\nabla \times \mathbf{A})$, so that in vector form, with $\mathbf{E} = -\nabla \phi$, and $\mathbf{B} = \nabla \times \mathbf{A}$, our equations of motion read:

$$ma = q \mathbf{E} + q \mathbf{v} \times \mathbf{B} \quad (1.153)$$
as always.

**Problem 1.24**

We have a function $L(x, \dot{x}, \ddot{x})$, and we want to vary w.r.t. $x$ to find the equations of motion. The action is:

$$S[x] = \int L(x, \dot{x}, \ddot{x}) \, dt \quad (1.154)$$

so that

$$\delta S \equiv S[x + \eta] - S[x] = \int (L(x + \eta, \dot{x} + \dot{\eta}, \ddot{x} + \ddot{\eta}) - L(x, \dot{x}, \ddot{x})) \, dt
\approx \int \left( \frac{\partial L}{\partial x} \eta + \frac{\partial L}{\partial \dot{x}} \dot{\eta} + \frac{\partial L}{\partial \ddot{x}} \ddot{x} \right) \, dt. \quad (1.155)$$

We can use integration by parts if we assume that both $\eta$ and $\dot{\eta}$ vanish at the endpoints of the trajectory — this is reasonable if we imagine specifying the initial and final velocity of the particle along with the initial and final position. After integration by parts, we have:

$$\delta S = \int \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{x}} \right) \right) \eta \, dt, \quad (1.156)$$

and for $\delta S = 0$ to hold for all perturbing trajectories $\eta$, we have:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{x}} \right) = 0. \quad (1.157)$$

For $L = \epsilon a^2 + \frac{1}{2} m v^2 - U(x)$, the above gives:

$$-U'(x) - m \ddot{x} + 2 \epsilon (\dddot{x}) s = 0. \quad (1.158)$$

In order for $\epsilon \dddot{x}$ to have units of force, $\epsilon$ must have units kg s$^2$.

**Problem 1.25**

The Lagrangian is the Legendre transform of the Hamiltonian, we are replacing $p$ with $\dot{x} = \frac{\partial H}{\partial p} = p/m + \alpha x$, so that

$$p = m (\dot{x} - \alpha x). \quad (1.159)$$
From the definition of the Legendre transform, we have:

\[
L = p \dot{x} - H = m (\dot{x} - \alpha x) \dot{x} - \left[ \frac{1}{2} m (\dot{x} - \alpha x)^2 + \alpha x m (\dot{x} - \alpha x) \right] \\
= \frac{1}{2} m (\dot{x} - \alpha x)^2 .
\]

The associated equations of motion are:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \frac{d}{dt} (m (\dot{x} - \alpha x)) + m \alpha (\dot{x} - \alpha x) = 0
\]

so that

\[
m \ddot{x} - m \alpha^2 x = 0 \quad \rightarrow \quad m \ddot{x} = m \alpha^2 x .
\]

(1.161)  
(1.162)
Chapter 2

Relativistic Mechanics

Problem 2.1

a. If we take the trigonometric functions to hyperbolic trigonometric functions, we can use the rotation result directly:

\[ \bar{x} = \cosh \eta x + \sinh \eta c t \]
\[ \bar{t} = \sinh \eta x + \cosh \eta c t. \]  

(2.1)

Then we have:

\[ d\bar{x}^2 = \cosh^2 \eta dx^2 + 2 \cosh \eta \sinh \eta dx \, c \, dt + \sinh^2 \eta c^2 dt^2 \]
\[ c \, d\bar{t} = \sinh^2 \eta dx^2 + 2 \cosh \eta \sinh \eta dx \, c \, dt + \cosh^2 \eta c^2 dt^2. \]  

(2.2)

Subtract the bottom from the top, and we have:

\[ d\bar{x}^2 - c^2 d\bar{t}^2 = dx^2 - c^2 dt^2 \]  

(2.3)

using \( \cosh^2 \eta - \sinh^2 \eta = 1 \), so the invariant requirement holds.

b. The origin of \( \bar{O} \) is at \( \bar{x} = 0 \), from which we learn via (2.1), that:

\[ \sinh \eta = -\frac{x}{c t} \cosh \eta, \]  

(2.4)

and the \( x \) location of the origin is \( x = v t \), so

\[ \sinh \eta = -\frac{v}{c} \cosh \eta. \]  

(2.5)

Squaring both sides, and replacing \( \sinh^2 \eta = \cosh^2 \eta - 1 \), we have

\[ \cosh^2 \eta - 1 = \frac{v^2}{c^2} \cosh^2 \eta \rightarrow \cosh \eta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \]  

(2.6)